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Biochemistry. — *"The function of phosphates in the dissimilation of the hexoses."* By A. J. KLUYVER and A. P. STRUYK.

(Communicated at the meeting of January 30, 1926).

§ 1. *Introduction.*

An experimental investigation on the so-called co-enzyme of alcoholic fermentation led us to a closer study of the numerous researches which have been made on the first phases of the dissimilation of the hexoses. In this connection it was particularly necessary to pay attention to the part phosphates play in these conversions.

It is to the fundamental study of HARDEN and YOUNG 1906 [1]), that we owe the insight that in alcoholic fermentation a close relation exists between the decomposition of the hexose molecule and a conversion of the mineral phosphates present in the fermentation medium into organic compounds. They showed that by the addition of mineral phosphates to a fermenting mixture of free zymase and sugar the rate of fermentation is noticeably increased, namely to such a degree that for every molecule of phosphate that passes into an organic compound one molecule of carbon dioxide and one molecule of alcohol are produced in excess.

These observations have been fully confirmed by the later investigations and a series of publications on the biochemical phosphorylation has been issued. In recent years the interest in this process has considerably increased namely after it was found that the phenomenon just mentioned is not at all limited to alcoholic fermentation of sugars. It were EMBDEN and LAQUER [2]) who demonstrated the formation of a hexosephosphoric ester in the sugar dissimilation in the muscular tissue of animals, whereas recently VIRTANEN [3]) conclusively proved that also in the sugar dissimilation by true lactic acid bacteria and propionic acid bacteria the production of a phosphoric ester occurs.

Notwithstanding the considerable amount of information that has already been gained on the biochemical phosphorylation, we still grope in the dark as to the meaning of this, at first sight rather surprising, process. In support of this we may just quote NEUBERG and KOBEL [4])

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[1]) A. HARDEN and W. J. YOUNG, Proc. Royal Soc. Ser. B. Vol. 77, p. 405, (1906).

[2]) G. EMBDEN und F. LAQUER, Zeitschr. f. physiol. Chemie Bd. 93, p. 94, (1914); ibid. Bd. 98, p. 181, (1917); ibid. Bd. 113, p. 1, (1921); G. EMBDEN und M. ZIMMERMANN, ibid. Bd. 141, p. 225, (1924).

[3]) A. I. VIRTANEN, Zeitschr. f. physiol. Chemie Bd. 138, p. 136, (1924); idem, Societas Scientiarum Fennica, Comment. Phys.-Math. II, 20, (1925).

[4]) C. NEUBERG und M. KOBEL, Biochem. Zeitschr. Bd. 166, p. 488, (1925).

who wrote in their only one month old publication: „Die Bedeutung der Phosphorylierung ist bis zum heutigen Tage nicht durchsichtig."

A closer examination of the abundantly available experimental material, together with our desire to look also at this part of the sugar dissimilation in the light of the newer ideas about the dissimilation processes in general, enabled us to design a scheme into which the different data fit and that fulfills the requirement of elucidating the whole of the observations.

Before exposing this scheme we will briefly describe how HARDEN and YOUNG interpreted the observed facts, interpretation which is almost generally accepted up to this day. [1])

§ 2. *Critical consideration of the prevailing theory.*

The fact that both YOUNG [2]) and VON LEBEDEW [3]), and also later investigators have succeeded in isolating a hexosediphosphoric ester from the fermenting mixture of free zymase, hexose and phosphate, together with the above mentioned quantitative relation between the production of alcohol and carbon dioxide and the disappearance of the free phosphates, led HARDEN and YOUNG [4]) to give the following equations for the course of alcoholic fermentation:

$$2C_6H_{12}O_6 + 2PO_4HR_2 = 2CO_2 + 2C_2H_6O + 2H_2O + C_6H_{10}O_4(PO_4R_2)_2 \quad (I)$$

$$C_6H_{10}O_4(PO_4R_2)_2 + 2H_2O = C_6H_{12}O_6 + 2PO_4HR_2 \quad (II)$$

As appears from these equations the two English investigators have formed the idea that simultaneously with the phosphoric esterification of one molecule of hexose, a second molecule of hexose is decomposed into carbon dioxide and alcohol. The produced hexosephosphoric ester, however, is decomposed in its turn and consequently the hexose then formed can anew take part in the first reaction. They base this view on the observation, that on the one hand the rate of fermentation falls to its original value when the added phosphate is bound, whereas on the other hand it has appeared, that against the end of the fermentation the added phosphate entirely returns into the fermentation medium.

It might be superfluous te demonstrate that this conception, which indeed explains in an acceptable way the observed course of the rate of fermentation, at closer examination proves to be highly unsatisfactory, because it is quite inexplicable, that the esterification of one molecule should cause a second molecule of hexose, which has not the least part

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[1]) MEYERHOF too gives this interpretation in his monograph: Chemical Dynamics of Life Phaenomena. Chicago, 1924.

[2]) W. J. YOUNG. Proc. Royal Soc. Ser. B. Vol. 81, p. 528, (1909); Biochem. Zeitschr. Bd. 33, p. 178, (1911).

[3]) A. VON LEBEDEW, Biochem. Zeitschr. Bd. 20, p. 114, (1909).

[4]) See for instance: A. HARDEN, Alcoholic Fermentation, 3rd Edition, London, 1923.

in this reaction, to be decomposed into carbon dioxide and alcohol. [1]) Moreover it is very unsatisfactory, that in this way the phosphorylation does not concur to the elucidation of alcoholic fermentation, but on the contrary the last is postulated as a condition for the esterification. The signification of the esterification remains in this way wrapped in mist. Finally the fact, that as yet it has not been explained why the isolated hexosediphosphoric ester is fermented at a perceptibly slower rate by zymase than free hexose in the same concentration, is an other strong evidence against the prevailing theory [2]).

§ 3. *Hypothesis on the function of phosphates in the sugar dissimilation.*

In drawing up a new scheme concerning the function of phosphates in the dissimilation of the hexoses, we have in the first place been guided by our desire to make the formation of phosphoric esters a logical link in the whole chain of transformations of the hexose molecule. As will appear later on, this aim was highly favoured by the recent investigations on the configuration of the hexoses.

The starting point of our hypothesis further was the incontestable fact, that the fermentative dissimilation of the hexoses does in all cases lead to the production of compounds of the C_3-series.

Considering the possibility of reducing the succeeding stages of dissimilation processes to a catalytic transference of hydrogen, which DONKER and one of us (Kl) [3]) exposed in a previous communication, it is highly probable, that the transformations of the C_6 compounds into C_3 compounds will also consist in an intramolecular oxydoreduction.

At that time we have refrained from giving a further explanation of the fact why exactly a H-atom of the fourth C-atom was activated by the acting protoplasm. We will now, however, submit this point to a closer examination.

It is self-evident to suppose, that the stated formation of a phosphoric ester does not accomplish the decomposition of the hexose in the mysterious way pointed out by HARDEN and YOUNG, but on the contrary in a far more direct way, on the understanding, that it is the very hexose molecule that enters into reaction with the phosphate, which is decomposed. Now it will be clear, that at least in first instance undoubtedly a hexose *mono*phosphoric

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[1]) The fact, that biochemical hydrolyses and consequently also the esterifications are characterized by a very small energetical effect, shows the deficiency of the parallel which MEYERHOF draws between the connection of these two reactions and the connection he proved to exist between the combustion of a part of the lactate in muscular tissue and the simultaneously occurring synthesis to glycogen of an other part of the lactate. See: O. MEYERHOF, Chemical Dynamics of Life Phaenomena, 1924, p. 56.

[2]) See among others: C. NEUBERG und M. KOBEL l. c.

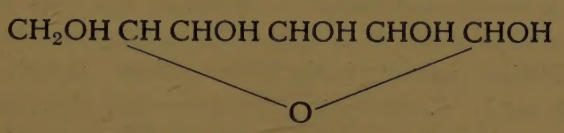
[3]) These Proceedings, Vol. 34, p. 237, (1925).

ester must be produced from this hexose molecule. In this connection we must immediately point to the fact, that the splendid investigation of ROBISON [1]) has shown that in a fermenting medium of free zymase, hexose and phosphate such an ester is actually produced.

Then, however, the question presents itself, at which of the 6 C-atoms such esterification will occur. To answer this question it is necessary to consider more closely the constitution of d-glucose.

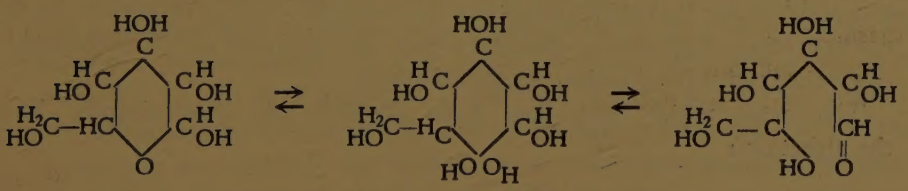
In recent years already several times it had been supposed, that d-glucose might have the configuration of an amylene oxide ring [2]). By the recent investigations of HAWORTH [3]) it has now been proved that this supposition was right.

So d-glucose has the following ringstructure:



As appears from the reducing properties of d-glucose this ring-structure will, in aqueous solution, have a strong tendency to pass by ring-opening, that is to say by hydrolysis, into a chain-structure.

With ARMSTRONG [4]) we may now conclude, that in this ring-opening intermediate formation of an aldehyde hydrate occurs, which may be rendered in a scheme as follows:



In realizing that reversely ring closure comes to a liberation of water by the reaction of two OH-groups which are bound to the first and fifth C-atom respectively, we may conclude that it must be one of these two apparently reactive OH-groups, which in enzymatical phosphorylation will bind the phosphate rest. That the phosphate rest should be bound to the first C-atom would be inconsistent with the fact, that the hexose *mono*-phosphoric ester, prepared by ROBISON, has strongly reducing properties.

Considering all this it seems probable to us, that the primarily formed

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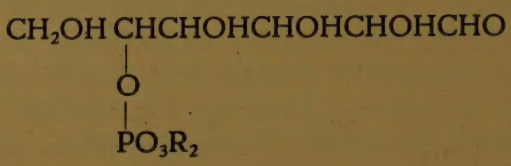
[1]) R. ROBISON, Biochem. Journ. Vol. 16, p. 809, (1922).

[2]) See for instance: J. BÖESEKEN, The Configuration of the Saccharides, Part. II, p. 28, Leyden.

[3]) W. N. HAWORTH, Nature, Vol. 116, p. 430, (1925).

[4]) E. F. ARMSTRONG, The Carbohydrates and the Glucosides. London, 1924, p. 8. It is true that ARMSTRONG here still gives the formerly accepted butylene oxide ring structure, but there seems to be no reason to suppose that this should cause any alteration in the above consideration.

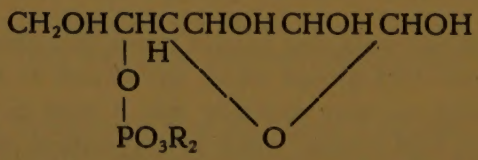
hexose monophosphoric ester will in first instance have the following formula :



The fact, that by the usual, in purely chemical way performed esterification, for instance of d-glucose by acetic acid, a penta-acetyl compound is produced, in which the acetyl-groups are bound to the OH-groups of the carbon atoms 1, 2, 3, 4 and 6, seems to us not at all inconsistent with our view. This is easily understood in realizing that this purely chemical esterification always occurs in an anhydrous medium and under addition of strong dehydrating agents (anhydrous sodium acetate, concentrated sulphuric acid, etc.). Contrary to the enzymatical phosphorylation which takes place in a diluted aqueous solution, in this case there can be no question of ringopening, that is to say of hydrolysis.

Meanwhile the above indicated ester will not be stable and in its turn will tend to pass into a ringstructure. As, however, the OH-group of the 5th C-atom is blockaded, it is self-evident that now the OH-group of the 4th C-atom will enter into reaction, in consequence of which a phosphoric ester of the butylene oxide ring is produced. This is the more readily acceptable since HAWORTH considers the butylene oxide structure to be the configuration of IRVINE's γ-glucose, which of late is also often mentioned in connection with biochemical transformations.

The ester which has now been formed will consequently have the structure :



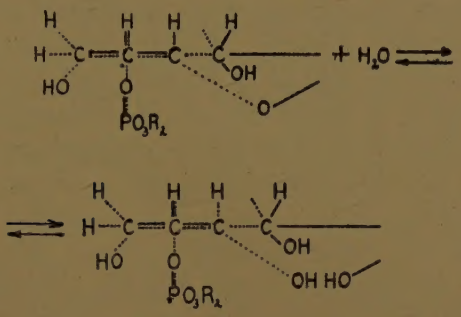
In our train of thought the transformation of the normal glucose into the last mentioned ester should now cause the H-atom at the fourth C-atom, under the influence of the hydrogen activating action of the protoplasm, to be activated to such a degree, that transference to the third C-atom, under simultaneous breaking up of the bond between these two C-atoms, takes place.

In other words, under the influence of the entered phosphate rest a dislocation in the molecule will have happened, which is indispensable for the fission of the C_6-compounds under the influence of the hydrogen activating protoplasm, so that, whereas the hydrogen activation by this last agent as such is not sufficient to achieve this effect in

normal glucose, this would only become possible by the additional effect of the action of the phosphate rest.

Perhaps the subjoined schematic representation of the action of the phosphate rest may serve to elucidate our view to some extent, although we fully recognize the relativity of its value.

If we simply reproduce by ..... a weakened, and by ═══ a strengthened bond between the atoms, then it is not excluded that the dislocation in the hereby more directly concerned part of the glucose molecule, effected by the phosphate rest, might be rendered as follows:



From the readiness, with which the phosphate rest can be split off, follows that the fifth C-atom in the ester is unsaturated to a certain degree, which will among others cause an increased occupation of the affinities of the 4th C-atom. This must lead to an activation of the H-atom, bound to this last C-atom, and a weakening of the bond between the fourth and the third C-atom; consequently this last becomes unsaturated and is so to say prepared to act as a hydrogen acceptor. If now to this the hydrogen activating action of the protoplasm is added, this will transfer the hydrogen atom from the fourth to the third C-atom, under simultaneous breaking up of the bond between the two C-atoms. The fourth C-atom, which has now become unsaturated will consequently also break up the C-O-C bond under hydration and thereby at both sides give rise to the formation of aldehyde groups.

Whereas we are fully aware of the fact, that hitherto we have entirely moved on hypothetical field, we will now trace, how far the available extensive experimental material agrees with our postulation.

§ 4. *Testing of our hypothesis to the available experimental material.*

From the foregoing section it will be clear that we have formed the idea, that in the dissimilation of glucose in first instance a glucose *mono*-phosphoric ester is produced, which then is decomposed into one molecule of glycerine aldehyde and one molecule of glycerine aldehyde phosphoric ester.

The whole course of the alcoholic fermentation might now be represented as follows:

$$C_6H_{12}O_6 + PO_4R_2H = C_6H_{11}O_5(PO_4R_2) + H_2O. \quad (I)$$

$$C_6H_{11}O_5(PO_4R_2) = C_3H_6O_3 + C_3H_5O_2(PO_4R_2). \quad (II)$$

$$C_3H_6O_3 = CO_2 + C_2H_6O. \quad (IIIa)$$

$$C_3H_5O_2(PO_4R_2) + H_2O = C_3H_6O_3 + PO_4R_2H. \quad (IIIb)$$

The $C_3H_6O_3$ produced in equation IIIb is afterwards also split up into carbon dioxide and alcohol according to equation IIIa. The course of the reaction IIIa is here left out of consideration ; for this we may refer to the before quoted publication of DONKER and one of us [1]).

Now our scheme does as well lend itself to the explanation of the course of the rate of alcoholic fermentation when phosphates are added to a sugar solution containing free zymase, as that proposed by HARDEN and YOUNG. Whereas in the scheme of HARDEN and YOUNG at the moment when all the phosphate is bound the rate of fermentation is controlled by the velocity of the hydrolysis of the hexose *di*phosphoric ester — this being the reaction which proceeds at the slowest rate — in our way of explanation the first mentioned rate is in a similar way controlled by the velocity of the hydrolysis of the triose phosphoric ester. Meanwhile from our scheme, as much as from that of the English investigators follows that by the addition of free mineral phosphates, according to our equations I, II and IIIa, in a certain period an extra amount of carbon dioxide and alcohol is formed, which is equivalent to the phosphate bound in the same period.

Consequently in the above discussed regard both ways of explanation are perfectly equivalent.

The scheme of HARDEN and YOUNG, however, fails to give an explanation of the convincingly proved fact, that *during the initial period of fermentation* the ratio of the esterified phosphate and the carbon dioxide produced is greater than 1 and not before the end of the phosphate period the ratio 1, observed by the said investigators, is reached [2]). As in our scheme the esterification precedes the fermentation and consequently a certain accumulation of hexose *mono*phosphate may occur before the production of carbon dioxide begins, the above mentioned phenomenon obtains an unconstrained explanation.

Further we have already brought into relief, that the production of the hexose *mono*phosphoric ester, inserted in our scheme, in the fermentation of glucose by free zymase in the presence of mineral phosphates, has been

[1]) With NEUBERG, who makes the same remark in connection with methylglyoxal, we are of opinion that the fact, that the rate of fermentation of the triosepreparations, which have up to now been examined, is smaller than that of the fermentable hexoses, need not in the least plead against the conception that trioses should intermediately occur in the fermentation of the hexoses. The phenomenon just mentioned may very probably be ascribed to the strong tendency of trioses to pass by hydration, polymerisation, or possibly formation of ring structures, into more stable compounds.

[2]) See the investigation of H. EULER und D. JOHANSSON, Zeitschr. f. physiol. Chemie. Bd. 85, p. 192, (1913). Compare also: O. MEYERHOF, ibid. Bd. 102, p. 204, (1918).

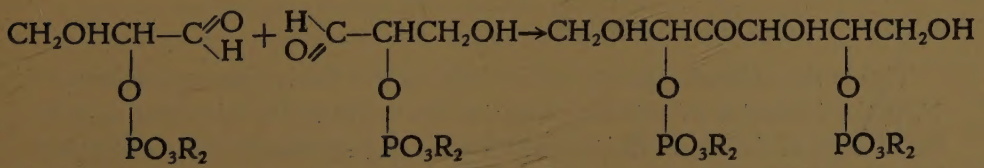
proved experimentally by ROBISON [1]). More important, however, is the fact, that this hexose *mono*phosphoric — contrary to the before mentioned hexose *di*phosphoric — ester is well fermentable and that with a velocity, which in accordance with our scheme practically agrees with the rate of the glucose fermentation, *enhanced* by the addition of phosphate.

Meanwhile the objection will be made against our above exposed view, that we have left out of account the already mentioned formation of hexose *di*phosphoric ester, conclusively proved by the investigations of YOUNG, VON LEBEDEW, NEUBERG a.o..

For the formation of this ester under the conditions chosen by YOUNG and ROBISON, whereby the fermentation proceeds in the presence of large quantities of mineral phosphates, we think however to be able to give an unconstrained explanation. We have seen, how in our way of representation the rate of fermentation is, on the basis of the results obtained by HARDEN and YOUNG, conditioned by the rate of the decomposition of the triose phosphoric ester, in last instance by the maximal capacity of the enzyme effecting the hydrolysis (the phosphatase). The remaining partial reactions must consequently proceed at a higher rate. Under these circumstances it is inevitable that the excessive addition of phosphates applied by YOUNG and ROBISON must lead to an accumulation of triose phosphoric ester.

Considering now the fact, that by the investigation of DONKER, VISSER T HOOFT and one of us (KI) [2]), it was convincingly proved, that the forced accumulation of acetaldehyde during the alcoholic fermentation causes a condensation of this substance to acetylmethylcarbinol (the carboligase action of NEUBERG) it is self-evident to suppose that the triose phosphoric ester accumulated under abnormal conditions is in quite a similar way condensed to hexose *di*phosphoric ester.

This transformation should then have to take place as follows:



Now in this connection we will not neglect to mention that already in 1907 the formation of a triose phosphoric ester in the considered fermentation medium was defended on experimental grounds by IWANOFF [3]). It is true that IWANOFF's observation has never been confirmed, but

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[1]) l. c.

[2]) A. J. KLUYVER, H. J. L. DONKER und F. VISSER 'T HOOFT, Biochem. Zeitschr. Bd. 161, p. 361, (1925).

[3]) L. IWANOFF, Zeitschr. f. physiol. Chemie. Bd. 50, p. 281, (1907); Centr. f. Bakter. IIe Abt. Bd. 24, p. 1, (1909).

apparently his adversaries have thereby lost out of sight, that IWANOFF isolated his ester in a fermentation mixture with far smaller phosphate concentration and which moreover was placed at a lower temperature. Under these circumstances it need not in the least be considered improbable that IWANOFF has indeed obtained a triose phosphoric ester.

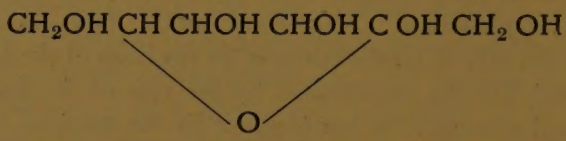
We will now examine more closely the consequences of our interpretation of the occurrence of the hexose *di*phosphoric ester.

Let us therefore consider how in our train of thought d-fructose will be fermented.

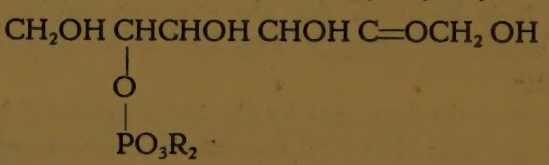
This hexose, with the chain structure

$$CH_2OH\ CHOH\ CHOH\ CHOH\ C{=}OCH_2OH$$

has the following ring structure according to the insight of the chemists:

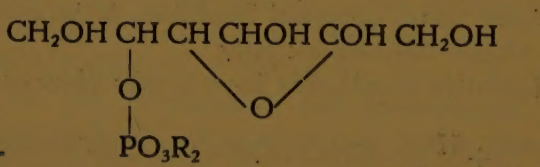


In analogy with what has been remarked in this respect for d-glucose we must take for granted that in the biochemical phosphorylation

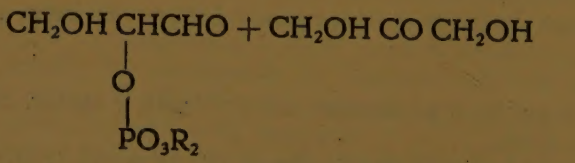


will be formed.

This ester however will only be an intermediate product, as by renewed ring closure it will pass into:



This is the monophosphoric ester of the so reactive γ-fructose of IRVINE. Now the formation of this ring explains unconstrainedly why it is exactly the H-atom of the fourth C-atom which after ring opening will by preference be activated by the acting protoplasm. This again will cause quite similarly the intramolecular transference of hydrogen from the fourth to the third C-atom by which will be produced:



Besides dioxyacetone the same glycerine aldehyde phosphoric ester will

be produced from d-fructose as well as from d-glucose, and from this ester under the indicated conditions again the same hexose *di*phosphoric ester will be formed.

Also for d-mannose we must accept a quite analogic course as for d-glucose, whereby again the same hexose *di*phosphoric ester must occur.

This inevitable consequence of our view now is perfectly in keeping with what already long ago has been experimentally ascertained by HARDEN and YOUNG [1]), but which hitherto had not been satisfactorily explained.

Resuming we may shortly state that the significance of the biochemical phosphorylation of the hexoses is to be found in the conversion of sugars into the so reactive γ-modifications.

Furthermore the given way of representation affords an unsought explanation of the contrast which in alcoholic fermentation in several respects appears to exist between glucose and mannose on the one side and fructose on the other side [2]). We need only consider that in reaction II the two first mentioned sugars produce glycerine aldehyde, fructose on the other hand produces dioxyacetone.

Meanwhile there is still another consequence of our view to be examined. We must namely consider the question, which sugar will be produced when phosphoric acid is split off from the hexose *di*phosphoric ester. Considering the constitution of the hexose *di*phosphoric ester as following from our theory it will be obvious, that a hexose will be formed, which will have the chain structure: $CH_2OH\ CHOH\ CHOH\ CO\ CHOH\ CH_2OH$, consequently a 3-ketohexose. What do now the experiments show? NEUBERG, FÄRBER, LEVITE and SCHWENK [3]) have convincingly proved that in the hydrolysis of the hexose *di*phosphoric ester with oxalic acid d-fructose is formed. Considering the inconsistency of the 3-ketohexoses and the readiness with which the C=O group by enolisation transmigrates in the molecule, it may not be bold to suppose that only during the preparation fructose is formed from the just mentioned 3-ketohexose. And we feel the sooner entitled to express this opinion since YOUNG has already stated long ago that the *solution* of the sugar split off from the ester has a ratio of reduction and optical rotation which does not agree with that of d-fructose.

Consequently neither in this respect does our conception appear to encounter difficulties. On the contrary it gives a ready explanation of the fact that also the hexose diphosphoric esters prepared from glucose and mannose respectively, on hydrolysis produce d-fructose.

Since further according to our view in the three hexoses mentioned the esterification attacks the molecules in those parts which are identical, as far as the position of the H-atoms and the OH groups in space is concerned, the produced triose phosphoric esters also must in stereo-

[1]) A. HARDEN und W. J. YOUNG, Proc. Royal Soc. Ser. B. Vol. 81, p. 336, (1909).

[2]) See: A. HARDEN, Alcoholic Fermentation, p. 120.

[3]) C. NEUBERG, E. FÄRBER, A. LEVITE und E. SCHWENK, Biochem. Zeitschr. Bd. 83, p. 244, (1917).

chemical respect be quite identical. Of course the same holds good for the hexose *di*phosphoric ester formed bij the condensation.

Finally we want to remark that our view also implies the fact, stated by ROBISON [1]), that the well fermentable hexose *mono*phosphoric ester he isolated is in no respect identical with the hexose *mono*phosphoric ester prepared by NEUBERG [2]) by partial hydrolysis of the hexose *di*-phosphoric ester. Perfectly in accordance with the anticipations of our view ROBISON obtained in chemical and enzymatical hydrolysis of his hexose *mono*phosphoric ester a dextro-rotatory reducing solution, from which he prepared glucosazone.

Also the divergent behaviour of the hexose-*di*phosphoric ester and of ROBISON's *mono*phosphoric ester towards phenylhydrazine is fully explained by the constitution of these esters accepted by us. The hexose-*mono*phosphoric acid osazone prepared from hexose-*di*phosphoric acid is, as may be expected from our scheme, *not* identical with the hexose-*mono*-phosphoric acid osazone from ROBISON's *mono*-compound.

§ 5. *Conclusion.*

In the foregoing we have developed a new theory on the function of phosphates in the dissimilation of the hexoses. This theory is, as far as the general course of the rate of alcoholic fermentation in presence of phosphates is concerned, equivalent to the current mode of explanation as formulated by HARDEN and YOUNG.

The remarkable situation now occurs, that the experiments one would be inclined to carry out in the first place in verification of our hypothesis have already been described in the extensive literature.

The following points may illustrate this:

1st. Our theory implies that in dissimilation the hexoses are in first instance transformed into a hexose *mono*phosphoric ester. This ester was indeed isolated by ROBISON.

2nd. Our theory implies that this hexose monophosphoric ester is fermented at a rate which agrees with the rate of fermentation of glucose, enhanced by the addition of phosphates. This was actually stated by ROBISON.

3rd. Our theory affords an explanation of the fact that during the initial period of fermentation the ratio of the esterified phosphate to the carbon dioxide produced is greater than 1, as has been proved by the experiments of EULER and JOHANSSON.

4th. Our theory concerning the way the hexose *di*phosphoric ester is formed implies that from d-glucose, d-fructose and d-mannose quite identical, also with regard to the stereochemical configuration, esters are produced. This was conclusively proved by HARDEN and YOUNG.

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[1]) l. c.

[2]) C. NEUBERG, Biochem. Zeitschr. Bd. 88, p. 432, (1918).

5th. Our theory implies that the hexose *mono*phosphoric ester isolated by ROBISON is not identical with the hexose *mono*phosphoric ester prepared by NEUBERG from the before mentioned hexose *di*phosphoric ester. This appears from the investigation of ROBISON. Also the behaviour of hexose-*di*- and of ROBISON's hexose-*mono*-phosphoric acid towards phenylhydrazine agrees very well with our view.

6th. Our theory implies that by hydrolysis of the hexose *di*phosphoric ester obtained from d-fructose and also from d-mannose a same hexose is split off, this being a 3-ketohexose, which we may suppose to pass easily into fructose. The isolation of d-fructose after hydrolysis of this ester by NEUBERG and his collaborators concurs with this view.

We wish to state expressly that the current view absolutely fails to elucidate these experimentally established facts, which on the other hand are logical consequences of our theory.

To this we may add, that the theory set forth in the above affords also an unconstrained explanation of the fact, stated by HARDEN and YOUNG, that the fermentation of d-glucose and d-mannose on the one hand and d-fructose on the other differ in many respects.

Finally in our theory the formation of phosphoric esters is for the first time made into a logical element in the whole of the dissimilation of the hexoses, because phosphates are the agent which converses these sugars into the so reactive γ-form.

Delft, January 1926.

Mathematics. — *"Projective and conformal invariants of half-symmetrical connections"*. By J. A. SCHOUTEN. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of December 19, 1925).

1. *The quantity of projective curvature.*

We consider a linear connection [1]) leaving transvections invariant:

$$\begin{aligned} \delta v^\nu &= dv^\nu + \Gamma^\nu_{\lambda\mu}\, v^\lambda\, dx^\mu \\ \delta w_\lambda &= dw_\lambda - \Gamma^\nu_{\lambda\mu}\, w_\nu\, dx^\mu \end{aligned} \quad \ldots\ldots\ldots \quad (1)$$

If

$$S^{\cdots\nu}_{\lambda\mu} = {}^1\!/_2\,(\Gamma^\nu_{\lambda\mu} - \Gamma^\nu_{\mu\lambda}) \quad \ldots\ldots\ldots \quad (2)$$

has the form

$$S^{\cdots\nu}_{\lambda\mu} = S_{[\lambda}\, A^\nu_{\lambda]}, \quad \ldots\ldots\ldots \quad (3)$$

the connection is called a halfsymmetrical one [2]).

First we take $S^{\cdots\nu}_{\lambda\mu}$ quite general and using the quantity of curvature $R^{\cdots\nu}_{\omega\mu\lambda}$ we form a quantity $P^{\cdots\nu}_{\omega\mu\lambda}$ in the same way as this is done in an affine connection [3])

$$P^{\cdots\nu}_{\omega\mu\lambda} = R^{\cdots\nu}_{\omega\mu\lambda} - 2\,P_{[\omega\mu]}\,A^\nu_\lambda + 2\,A^\nu_{[\omega}\,P_{\mu]\lambda} \quad \ldots\ldots \quad (4)$$

$$P_{\mu\lambda} = -\frac{1}{n^2-1}\,(n\,R_{\mu\lambda} + R_{\lambda\mu}) \quad \ldots\ldots\ldots \quad (5)$$

$$R_{\mu\lambda} = R^{\cdots\nu}_{\nu\mu\lambda} \quad \ldots\ldots\ldots \quad (6)$$

Then the following theorems hold:

I. *In a halfsymmetrical connection $P^{\cdots\nu}_{\omega\mu\lambda}$ is invariant with all geodesical transformations of the $\Gamma^\nu_{\gamma\mu}$, which leave S_ν invariant.*

II. *If $P^{\cdots\nu}_{\omega\mu\lambda}$ is invariant with all transformations wich leave $S^{\cdots\nu}_{\lambda\mu}$ invariant, the connection is halfsymmetrical.*

III. *A halfsymmetrical connection can then and only then be transformed by a geodesical transformation of the $\Gamma^\nu_{\lambda\mu}$, which leaves S_λ invariant, into a displacement with a zero quantity of curvature, if $P^{\cdots\nu}_{\omega\mu\lambda}$ is zero.*

We have some remarkable identities. For a halfsymmetrical connection holds:

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[1]) Der Ricci Kalkül, SPRINGER 1924, p. 67.
[2]) R. K. p. 69.
[3]) R. K. p. 131.

$$\left.\begin{aligned}\nabla_\nu P_{\omega\mu\lambda}^{\cdots\nu} = &-2(n-2)\nabla_{[\omega} P_{\mu]\lambda} + \\ &+\frac{12}{n+1} S_{[\omega} R_{\mu]} - \frac{12}{n+1} S_{[\lambda} \nabla_\omega S_{\mu]} - 4 S_{[\omega} R_{\mu]\lambda} + 2 S_\nu R_{\omega\mu\lambda}^{\cdots\nu}\end{aligned}\right\} . \quad (7)$$

For an affine connection this identity passes into:

$$\nabla_\nu P_{\omega\lambda\mu}^{\cdots\nu} = -2(n-2)\nabla_{[\omega} P_{\mu]\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If $P_{\omega\mu\lambda}^{\cdots\nu} = 0$ the conditions

$$\nabla_{[\omega} S_{\mu]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\nabla_{[\omega} P_{\mu]\lambda} = 2 S_{[\omega} P_{\mu]\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

hold for a halfsymmetrical connection.

2. *The quantity of conformal curvature.*

We consider a metrical connection leaving transvections invariant, and we suppose $S_{\lambda\mu}^{\cdots\nu} \neq 0$ [1]). The $\Gamma_{\lambda\mu}^{\nu}$ are given by the equation [2]):

$$\Gamma_{\lambda\mu}^{\nu} = \left\{ \begin{matrix} \lambda\,\mu \\ \nu \end{matrix} \right\} + S_{\lambda\mu}^{\cdots\nu} - 2 S_{\cdot(\lambda\mu)}^{\nu} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Using the quantity of curvature, which we will call $K_{\omega\mu\lambda}^{\cdots\nu}$, we form a quantity $C_{\omega\mu\lambda}^{\cdots\nu}$ in a perfectly analogous manner as this is done in a RIEMANNIAN connection:

$$C_{\omega\mu\lambda\nu} = K_{\omega\mu\lambda\nu} - \frac{4}{n-2} g_{[\omega[\lambda} L_{\mu]\nu]} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$L_{\mu\lambda} = -K_{\mu\lambda} + \frac{1}{2(n-1)} K g_{\mu\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$K_{\mu\lambda} = K_{\nu\mu\lambda}^{\cdots\nu} \quad ; \quad K = K_{\mu}^{\cdot\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Then the following theorems hold:

I. *In a halfsymmetrical connection $C_{\omega\mu\lambda}^{\cdots\nu}$ is invariant with all conformal transformations of $g_{\lambda\mu}$.*

II. *If $C_{\omega\mu\lambda}^{\cdots\nu}$ is invariant with all conformal transformations of $g_{\lambda\mu}$, the connection is halfsymmetrical.*

III. *A halfsymmetrical connection can then and only then be transformed by a conformal transformation of $g_{\lambda\mu}$, which leaves S_λ invariant, in a connection with a zero quantity of curvature, if $C_{\omega\mu\lambda}^{\cdots\nu}$ is zero.*

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[1]) R. K. p. 72.
[2]) R. K. p. 73, equation (53*b*).
[3]) R. K. p. 170.

Here also some remarkable identities exist.
For a halfsymmetrical connection we have

$$\left.\begin{aligned}\nabla_\nu C_{\omega\mu\lambda}^{\cdots\nu} = -2\frac{n-3}{n-2}\nabla_{[\omega} L_{\mu]\lambda} - S_{[\omega} R_{\mu]\lambda} + \\ + 2S_\nu R_{\omega\mu\lambda}^{\cdots\nu} + \frac{4}{n-2} g_{\lambda[\omega} G_{\mu]\nu} S^\nu\end{aligned}\right\} . \quad (15)$$

$$G_{\mu\nu} = -L_{\mu\lambda} + L g_{\mu\lambda} \quad ; \quad L = L_\mu^{\cdot\mu} \; . \; . \; . \; . \; . \quad (16)$$

$$\nabla^\alpha G_{\mu\alpha} = 2 S^\alpha G_{\alpha\mu} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (17)$$

For a RIEMANNIAN connection this identity passes into:

$$\nabla_\nu C_{\omega\lambda\mu}^{\cdots\nu} = -2\frac{n-3}{n-2}\nabla_{[\omega} L_{\mu]\lambda} \; . \; . \; . \; . \; . \; . \quad (18)$$

If $C_{\omega\mu\lambda}^{\cdots\nu} = 0$, the relations

$$L_{[\mu\lambda]} = -(n-2)\nabla_{[\mu} S_{\lambda]} \; . \; . \; . \; . \; . \; . \; . \quad (19)$$

$$\nabla_{[\omega} L_{\mu]\nu} = 2 S_{[\omega} L_{\mu]\nu} \; . \; . \; . \; . \; . \; . \; . \quad (20)$$

hold for a halfsymmetrical connection.

Microbiology. — *"On the Disinfection of the Bacteriophagus of Bacillus Danicus."* By N. L. SÖHNGEN and A. GRIJNS. (Communicated by Prof. A. H. BLAAUW.)

(Communicated at the meeting of October 31, 1925).

D'HERELLE may have advocated his view concerning the nature of the bacteriophage as a living organism ever so well, still there are many researchers, who hold a quite different opinion. Most notable amongst those are SEIFFERT, OTTO, MUNTER, DA COSTA CRUZ, DOERR, BORDET, CIUCA and BAIL.

When a bacteriophage is considered to be a living organism, the question arises whether the individual differences in regard to a certain property, that may exist between the bacteriophages of the same stock, are not demonstrable.

As known, several researchers, as CHICK, REICHENBACH, MADSEN, and NIJMAN, and above all EIJKMAN [1]) have examined the shape of the "curve of survivors" of various species of bacteria, when they were killed off either by thermal means or by disinfection. Their object also was to ascertain whether the curve of survivors corresponded with the frequency-curve of QUETELET-GALTON.

For Bacilli coli, killed off by a harmful temperature, and by the addition of phenol EIJKMAN [2]) found approximately a slanting individual variation-curve, while spores died in accordance with the formula of the unimolecular reaction.

We deemed it interesting to endeavour to construct also a curve of survivors of a bacteriophage by heating to a noxious temperature, and selected for our purpose a bacteriophage of Bacillus Danicus, an organism first described by LÖHNIS and WESTERMANN [3]). A short description may follow here.

Its shape and size are very variable. In broth they appear as strikingly large, paired rods about 2 μ in breadth and 8 μ in length; in a mannite solution they are rather diplo-, and streptococci. In both media distinct granulation. Immotile. After some days intense spore formation; globular and oval spores. Growth on meat-agar luxuriant, light yellow; in broth, in extracts of fish, yeast, and malt good. Sugars are not fermented. Also on non-nitrogenous azotobacterplates growth can be stated. N-assimilation moderate. On a plate with 5 % saccharose a distinctly visible ring of laevulan is formed round the colonies. Gelatin is liquified and ureum is split up. Acid-production from sugars is not demonstrable on a plate with litmus. Contains katalase, but no lipase. Is obligate aerobic.

1) Biochem. Zeitschr. 11. 12.
2) Folia Microbiologica 1. 359.
3) Centralbl. f. Bakt. Abt. II. bd. 22. 251.

The bateriophage of Bac. Danicus was isolated from the ground by the usual method. Its qualities directly appeared to be good, as it is competent to dissolve completely bacteria some days old, and moreover, never renders a secundary culture.

For the determination of the bacteriophage-concentration we had the alternative of plating or diluting. With the first method the number of plages (clear spots) was determined, that appears after spreading a known quantity of liquid of a certain dilution on a plate, smeared beforehand with bacteria. By the second method we could determine the so-called lysis-exponent, i.e. a series of dilutions is made of the liquid under examination, so that the titer is 10^{-1}, 10^{-2}, 10^{-3}, etc. The number of the last tube in which lysis still appears after inoculation with bacteria, is the lysis-index.

Accurate examination, such as for the tracing of a curve of survivors, cannot be made by the second method. This will be readily understood, if we reflect that if the last tube of the series still contains in a volume of 10 cc. of the fluid, say, 4 bacteriophages, it will be a mere accident if lysis still occurs, when 1 cc. is transplanted into the next-following tube. So we can never make sure whether there were in the original fluid 100.000 or 10.000 bacteriophages per cc. When the titers are made to differ less, the dilution errors become too great, while when we try to avoid fortuity by transplanting the last tube but one into other tubes, the procedure becomes too laborious for many observations.

DOERR and GRÜNINGER [1]) object to the plating method that the number of plages varies with the density of the layer of bacteria, and likewise with the degree of moistness of the agar-surface.

D'HERELLE arrived at the conclusion that the number of plages remained rather constant in spite of the increasing density of the suspension, whereas GRATIA observed, that by denser suspensions the number increases up to a certain limit and then again decreases.

BECKERICH and HAUDUROY [2]) compared the two counting methods and became convinced that the plating method was preferable for virulent bacteriophages, and the diluting method for weak ones, and that both methods were suitable for mediocre bacteriophages.

In conclusion, also DOERR and ZDANSKY [3]) have investigated the subject. They found that the diluting method yields still a positive result with a 100—1000 times stronger dilution than is used with the plating method; 100 times when the bacteria were put on the plate before the bacteriophage and 1000 times when the reverse took place, apart from the virulence of the bacteriophage.

However all this may be, for the virulent bacteriophage of Bac. Danicus the plating method appeared to yield perfectly reliable results, as was borne

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[1]) Zeitschr. f. Hyg. u. Inf. Krkh. 97. 209.
[2]) C. R. d. l. Soc. de Biol. 86. 165.
[3]) Zeitschr. f. Hyg. u. Inf, Krkh. 100. 78.

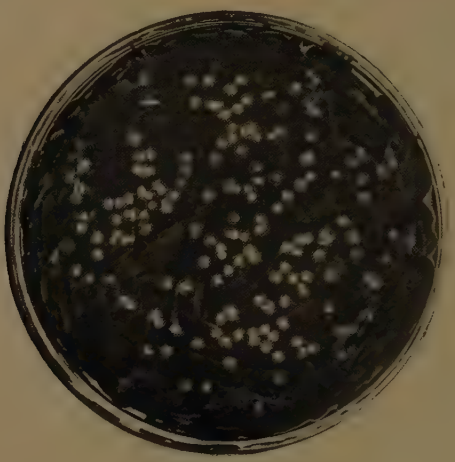

Fig. A.

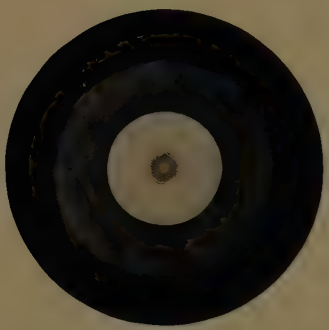

Fig. B.

Fig. C.

Fig. A.

$^1/_2 \times$ natural size; transmitted light.

Culture-plate, 18 hours old, used in the present investigation for counting the number of bacteriophages. On the plate a coating of bacterial material of Bacillus Danicus with plages of its bacteriophage.

Fig. B.

$2 \times$ natural size; transmitted light.

Plage, 4 days old. In the clear central field all bacteria have been dissolved; a coating has arisen in the centre, not composed of living material. In the partly transparent, circular zône only a part of the bacteria have been dissolved.

Fig. C.

$2.2 \times$ natural size; transmitted light.

Plage, 12 days old, with circular elevations in the agar-surface, sometimes covered with precipitates. Again the coating in the centre, which is not a secondary growth of bacteria. The partly transparent zône is still visible, but is less sharply outlined against the bacterial coating, which is becoming thinner and more transparent.

out by experiments, but may as well be inferred from the values recorded below.

A photo (A) of such a counting plate accompanies this article, the plages are of the bacteriophage of Bac. Danicus. It will be observed that the layer of bacteria is not evenly distributed over the plate. First the plates were dried over calcium chlorid, and then the dried suspension was once more smeared evenly over the plate. Later on this appeared to be unnecessary and even not desirable in view of the possible reproduction of the bacteriophage during that interval of time. The plages, going to the thinner layers can equally well be counted, while the number of late arrivals, which will occur in bacterial colonies, do not vitiate the result, if only the counting be done accurately. We also adjoin the photo's B and C of plages of different ages. (See the explanation of the plates).

The titer of the undiluted bacteriophage appeared to vary considerably from 13 to 340.10^7. We always got an impression that with this bacteriophage the titer is determined by the number of bacteria inoculated. The flasks were not more turbid than they were directly after the inoculation, at 36° lysis set in after $1\frac{1}{2}$ hours. Of course matters differ with a less virulent bacteriophage.

The disinfection experiment was conducted as follows: a flat, very thin-walled, glass shakingbottle, with a narrow neck, is fastened to a shakingapparatus by which the bottle is swung to and fro about 70 times per minute. The bottle is immersed up to the neck in a large water-bath, the water being kept at a constant temperature of 0.1° and stirred automatically.

The bottle contains 99 cc. 0.6 % NaCl-solution, to which 1 % broth has been added. At the beginning of the experiment, when the fluid in the bottle had attained the required temperature, 1 cc. of undiluted bacteriophage was added.

The level of the liquid ad 100 cc. is not higher than one third of the height of the bottle. As the warm water outside the bottle reaches up to the neck, and the shaking-velocity is just failing to cause the water in the bottle to splash, there is no chance whatever of experimental errors that would otherwise be entailed by droplets in the neck of the bottle that have escaped the heating. The bottle is wool closed by a cotton plug.

As a rule with each observation 0.5 cc. was taken from the quietly swinging bottle and was put immediately into a flask with 49.5 cc. of the fluid contained in the shaking bottle, so that we obtained a dilution 1 : 100.

We obtained 1 : 10 by adding 5 to 45 cc. Although in the dilution-technique of the bacteriophage frequent use is made of test-tubes filled with 9 cc. of the fluid, to which subsequently 1 cc. is added, we gave the preference to flasks with more liquid, by which we attained our purpose sooner, and the bacteriophage could be better divided by brisk shaking.

The final dilution was such that 0.1, 0.2, 0.3, or sometimes 0.5 cc. spread over the plate, covered beforehand with bacteria (meat-agar of $1\frac{1}{2}$ % with 1 % glucose), yielded a fair number of plages, viz. of about 50—500. Irrespective of the time it takes we can count with the counting apparatus more than 1000 plages per plate.

To find our bearings we started with the following observations. At 16° we ascertained whether the bacteriophage during its sojourn in the shaking-bottle with physiological salt-water with 1 % broth was injured. This appeared not to be so within three hours.

TABLE 1 (16°).

Time	Plages per plate	Dilution
0′	226	50.000
1′	225	id.
15′	193	id.
30′	224	id.
60′	223	id.
120′	238	id.
180′	221	id.

Neither was any injury visible at 38.5° in 3 hours :

TABLE 2 (38.5°).

Time	Plages per plate	Dilution
0′	85	100.000
1′	86	id.
15′	88	id.
30′	96	id.
60′	95	id.
120′	84	id.
180′	94	id.

At 41.5° it is doubtful after 3 hours :

TABLE 3 (41.5°).

Time	Plages per plate	Dilution
0′	80	100.000
2′	94	id.
15′	88	id.
30′	97	id.
60′	90	id.
120′	94	id
180′	66	id

At 45° disinfection is distinctly observable :

TABLE 4 (45°).

Time	Plages per plate	Living %	Dilution
0′	598	100.—	50.000
1′	590	98.3	id.
15′	533	89.—	id.
30′	473	79.—	id.
45′	462	77.—	id.
60′	455	75.7	id.
75′	442	73.3	id.

At 60° the velocity of disinfection was very great :

TABLE 5 (60°).

Time	Plages per plate	Dilution	Total per cc
0′	220	50.000	11.10^6
1′	—	—	—
5′	529	5	2645
10′	103	5	515
15′	51	5	255
25′	29	5	145

Ultimately 50° appeared to be a fit temperature for a disinfection experiment.

The subjoined table (6) expresses the values obtained. The number of plages per plate, multiplied by the dilution factor f_v, expresses the number of bacteriophages per cc. in the shaking bottle ; as will be seen the concentration in the incipient lysate is still 100 times higher. The other numbers have been calculated after the formula of the unimolecular reaction, viz.

$$K = \frac{1}{0.434 \times t} \log \frac{a}{a-x}.$$

In the 5th column the survivals have been expressed in percentages, in the 6th the logarithms of these numbers, in other words *log (a—x)* and in the 7th column the values K calculated from the formula.

TABLE 6 (50°).

Time	Pl. p. pl.	f_v	Pl. total in 10^6	% Survivors	*log.*	*K*
0′	680	50.000	34.—	100.—	2.—	—
1′	635	id.	31.8	93.4	1.97	0.069
3′	585	id.	29.3	86.—	1.93	0.054
5′	569	id.	28.5	83.7	1.92	0.037
7′	521	id.	26.1	76.6	1.88	0.040
10′	752	33.333	25.1	73.8	1.87	0.030
13′.30″	683	id.	23.8	69.9	1.84	0.028
17′	668	id.	22.3	65.5	1.82	0.024
21′	621	id.	20.7	60.9	1.78	0.024
25′	541	id.	18.0	53.—	1.72	0.027
30′	488	id.	16.3	47.9	1.68	0.025
40′	419	id.	14.0	41.1	1.61	0.023
50′	378	id.	12.6	34.1	1.53	0.022
60′	971	10.000	9.7	28.6	1.46	0.021
75′	646	id.	6.5	19.—	1.28	0.022
90′	663	id.	6.6	19.5	1.29	0.022
105′	1014	5.000	5.1	14.9	1.17	0.018
120′	1756	2.500	4.4	12.9	1.11	0.017

Graph. 1 gives a clearer survey. The continuous line shows the percentages of survivors; the dotted line shows how the values of *log (a—x)* vary with the time. Surely, it matters little whether *log (a—x)* or $\log \frac{a}{a-x}$ is plotted, when only the question is concerned whether the curve is a straight line or not. The lengths of the ordinates above the dotted line show the values of the last named expression.

We selected a temperature that would not kill off the bacteriophages too quickly and we have taken care to make at the outset a good number of observations, the typical part of the individual variation-curve being the very first portion.

It will be seen both from the table and from the graph that the disinfection is not at all slow; on the contrary the velocity is at the beginning even greater than later on. On the whole the logarithmic line is fairly approximated, still there is a distinct "go" in the reaction-constant, which diminishes first at a quick rate and afterwards more slowly.

For a better examination of the slowing of the reaction velocity, two experiments were now made at 55°, in which, however, the initial concentration in the shaking-bottle A was more than 100 times higher than in

B, viz. resp. 20.100.000 and 182.000 per cc. This enabled us to realize the relation between reaction-velocity and concentration of the bacteriophage.

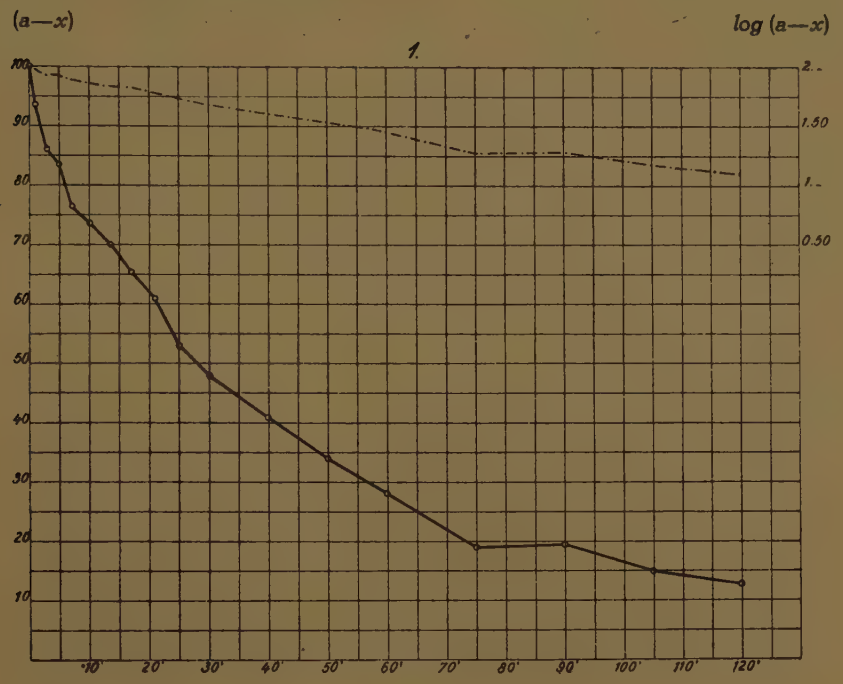

Fig. 1.

Table 7 and graph 2 give the values found. Of experiment B the table gives only the column of the reaction-constants (See table 7).

From this it is evident that the retrogression of the reaction-constants proceeds also in the following stages of the disinfection and ultimately becomes extremely small, the percentage of survivors then being 0.6—0.027.

TABLE 7 (55°).

Time	Pl. p. pl.	f_v	Total pl. in 10^3	% Surviv.	*log.*	K_A	K_B
0′	201	100.000	20.100	100.—	4.—	—	—
2′	123	50.000	6.150	30.6	3.49	0.60	0.55
5′	51	id.	2.550	12.7	3.10	0.41	0.39
10′	97	10.000	970	4.3	2.63	0.32	0.28
20′	60	5.000	300	1.5	2.18	0.21	0.19
30′	49	2.500	122.5	0.61	1.79	0.16	0.18
40′	46	1.000	46	0.23	1.36	0.15	0.14
50′	45	500	22.5	0.11	1.04	0.14	0.13
75′	541	10	5.4	0.027	0.43	0.11	0.11

It also appears that the reaction-constant is not affected perceptibly even by a hundred times lower bacteriophage-concentration.

The question may reasonably be asked whether the reaction-velocity is perhaps retarded so abnormally by the dead bacteriophages or by the reaction products. To make sure of this a new experiment was carried out

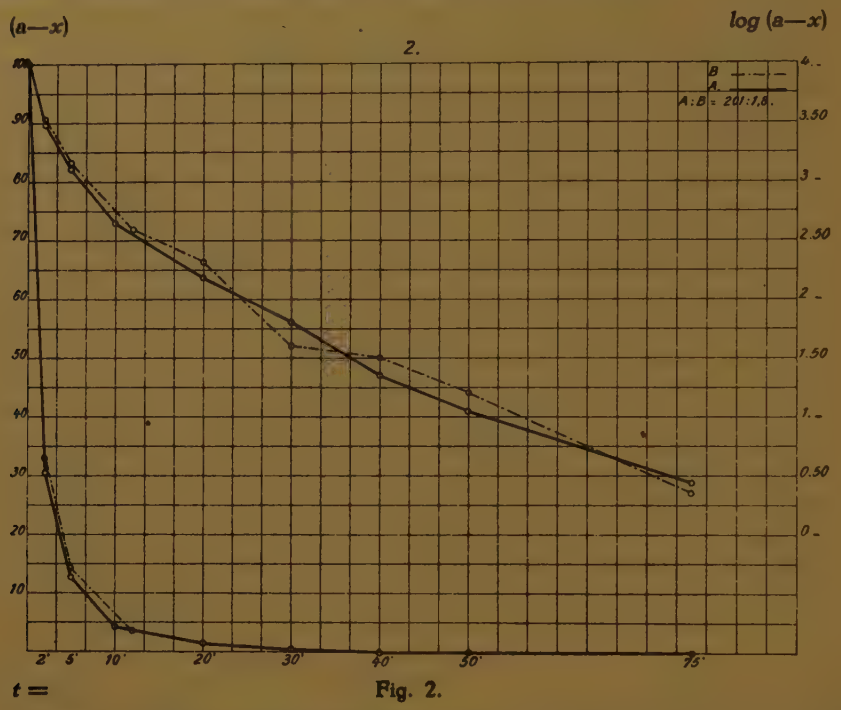

Fig. 2.

in exactly the same manner as the preceding one. First, however, 1 cc. of the physiological salt-solution was substituted in the shaking-bottle by 1 cc. of an undiluted lysate, which had been completely killed off by heating, so that at the initial stage of the experiment the dead material present exceeded 100 times the living material that had been added. Table 8 gives

TABLE 8 (55°).

Time	K_A	K_B	K_D
2′	0.60	0.55	0.63
5′	0.41	0.39	0.45
10′	0.32	0.28	0.27
20′	0.21	0.19	0.21
30′	0.16	0.18	0.16
40′	0.15	0.14	0.14
50′	0.14	0.13	0.13
75′	0.11	0.11	0.11

only the reaction-constants K_D, and beside them those of the preceding constants is given here either. The influence of the dead material is experiment for purposes of comparison.

We see, then, that no explanation of the retrogression of the reaction- practically nihil.

Lastly we considered the question what role could be played by the protein that had been added at 1 % (counted as broth) to the salt-water. To form some idea about it we made two more experiments ; in one medium there was 10 %, in the other none at all. It should be remembered, however, that even the highly diluted bacteriophage always carries along with it a quantum of protein as well from the broth used for the bacteria as from the dissolved bacteria themselves. Relatively the same excess remains with respect to the bacteriophage.

We first ascertained whether in such an approximately protein-free medium the bacteriophage was not injured already at room-temperature. That fear proved needless, at least for an interval of 3.5 hours.

TABLE 9 (15°).

Time	Plages per plate	f_v
0′	530	500
2′	504	500
60′	521	500
210′	525	500

Below this the reader will find the reaction-constants obtained, and percentages of survivors ; in graph 3 all data have been comprised. K_E are the reaction-constants for 10 % broth and K_V for protein-free.

So we see that the rate of reaction is indeed increased to such an extent

TABLE 10 (55°).

Time	% Surv. E	% Surv. V	K_E	K_V
0′	100.—	100.—	—	—
2′	20.2	13.5	0.79	1.—
5′	5.3	4.7	0.59	0.61
10′	3.3	1.—	0.34	0.46
20′	2.—	0.27	0.20	0.27
30′	0.91	0.041	0.16	0.26
40′	0.52	0.007	0.13	0.24
50′	0.32	—	0.12	—
75′	0.038	—	0.11	—

that without protein as little is left in 30′ as with protein in 75′, but that the abnormal decrease of velocity remains.

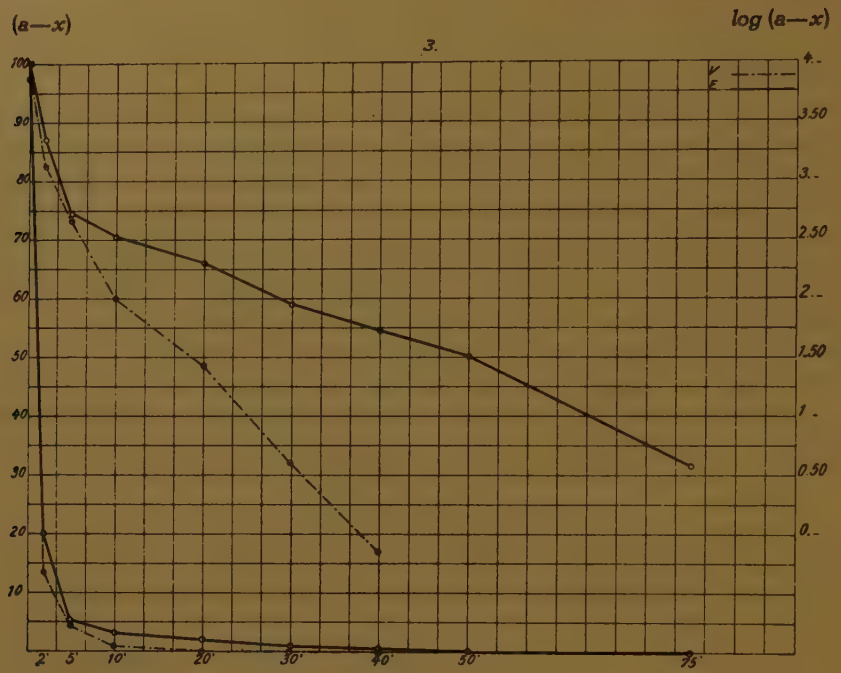

Fig. 3.

The possibility had still to be considered that this retardation might have its origin in the lysate, that might be composed of bacteriophages of differing resistance, as in transplanting a culture we never started from one and the same plage, so we never had made any selection.

To test this point further we inoculated one plage from an ordinary counting-plate, of which the planted bacteriophage had resisted during 50′ a temperature of 55°. The next day a disinfection-experiment (A) was made with it and on the day following another subculture (B). (Both

TABLE 11 (55°).

Time	% Surv. *A*	% Surv. *B*	K_A	K_B
0′	100.—	100.—	—	—
2′30″	7.7	(2′) 16.2	1.021	0.897
5′	1.3	5.6	0.874	0.575
10′	0.11	2.1	0.690	0.386
20′	0.03	0.29	—	0.127
30′	—	0.09	—	0.102
40′	—	0.02	—	0.093

in a protein-free medium.) The result are recorded in table 11 and in graph 4.

So the remarkable fact presents itself that the bacteriophage that has been subcultured once and has resisted for 50′ a temperature of 55°, is more affected by heating than in the normal circumstances, and that by repeated subcultures the normal sensitivity returns. The shape of the curve *log (a—x)* is essentially unaltered, which proves that, also when the bacteriophage is not previously heated, i.e. not selected, the decrease of the reaction-constants is not due to a combination of bacteriophage-types of different resistance. The recurrence of the normal sensitivity is a very remarkable and essential point, for which only life can be held responsible.

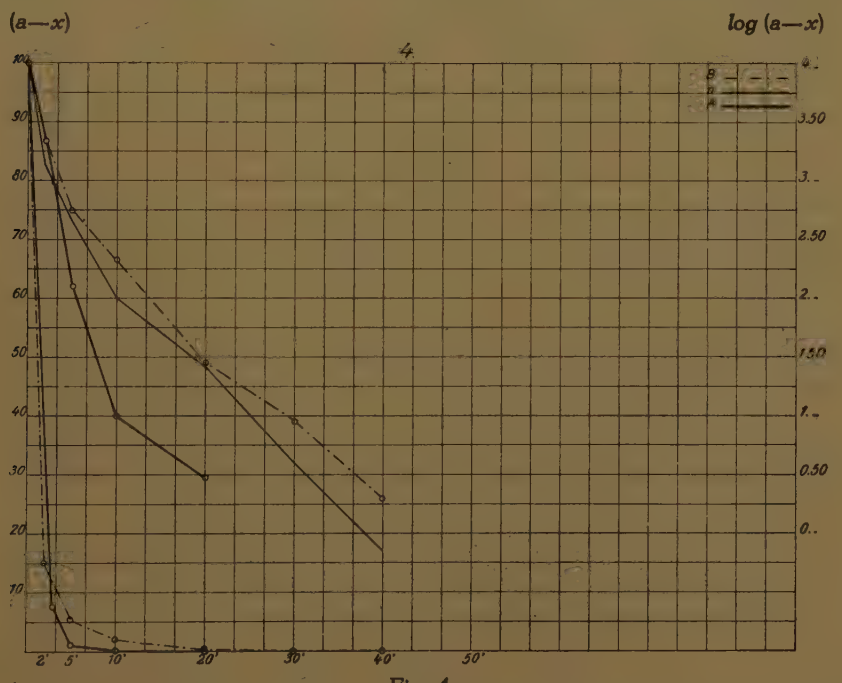

Fig. 4.

In order to ascertain whether the decreases of the reaction-constants also reveal themselves with another method of disinfection we conducted first of all a disinfection experiment at room-temperature in the usual way, but with addition of so much $\frac{N}{10}$ H_2SO_4, that in the saltwater of the shaking-bottle the p_H was 3.83.

Table 12 and graph 5 express the progress of the disinfection.

This curve shows that down to the end there is an acceleration in the disinfecting process.

It remains for us still to record an experiment on disinfection by radiation with ultraviolet rays. Undiluted lysate was put in a quartz erlenmeyer flask, attached to a shaking-apparatus right over a quartz-lamp. The distance from the bottom of the vessel to the lamp (220 V. alternating

TABLE 12 (15°, $p_H = 3.83$).

Time	Pl. p. pl.	f_v	Total in 10^3	% Surv.	*log.*	*K*
0′	72	1.000	72	100.—	2.—	—
5′	69	id.	69	95.9	1.98	0.009
10′	64	id.	64	88.9	1.95	0.012
20′	115	500	57.5	80.—	1.90	0.012
40′	90	500	45.—	62.5	1.80	0.012
60′	248	100	24.8	34.4	1.54	0.017
80′	284	50	14.2	20.—	1.30	0.020
120′	574	10	5.7	8.—	0.90	0.021
162′	810	2	1.6	2.25	0.35	0.021

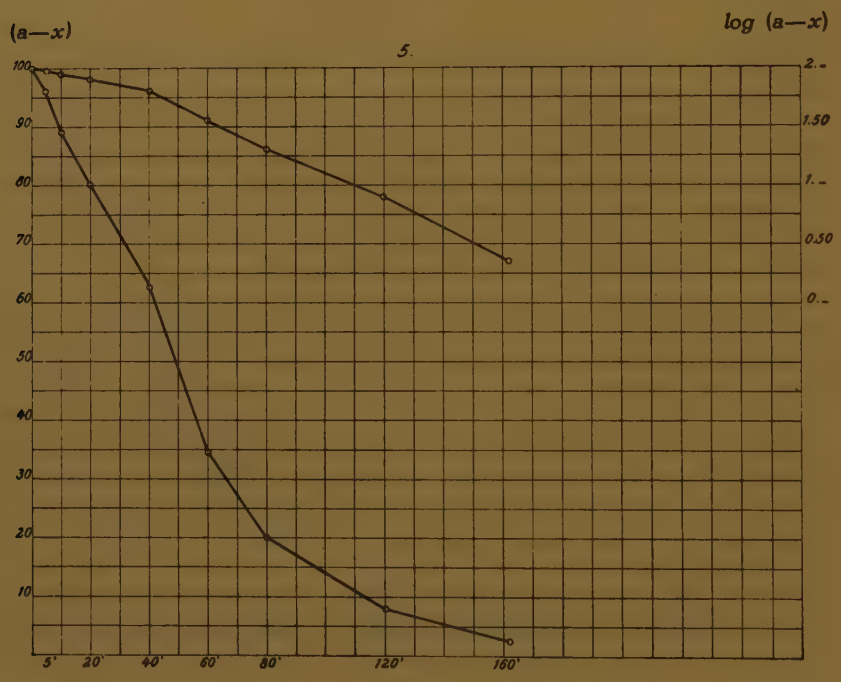

Fig. 5.

current) was 189 c.m. Prior to the radiation we allowed the lamp to burn 15′ to render the energy-radiation as constant as could be.

The values are combined in table 13 and graph 6.

We see, then, also here that initially the reaction-constants are rather constant and that subsequently there is a slight gradual rise.

What are we really doing when we kill off the bacteriophage by so-called disinfection ? Is the bacteriophage actually destroyed when no more plages come forth when sown ?

TABLE 13.

Time	Pl. p. pl.	f_v	Total in 10^6	% Surv.	$log + 1$.	K
0′	307	10^7	3070	100.—	3.—	—
4′	277	id.	2770	90.2	2.96	0.023
8′	255	id.	2550	83.—	2.92	0.023
12′	213	id.	2130	69.4	2.84	0.029
20′	178	id.	1780	58.—	2.76	0.016
28′	150	id.	1500	49.—	2.69	0.025
37′	—	—	—	—	—	—
48′	186	$\frac{1}{3}.10^7$	620	20.—	2.30	0.034
60′	121	id.	400	13.—	2.11	0.035
72′	54	id.	180	5.8	1.76	0.039
84′	61	10^6	61	2.—	1.30	0.046
96′	27	id.	27	0.88	0.94	0.048
120′	5	id.	5	0.16	0.20	0.053

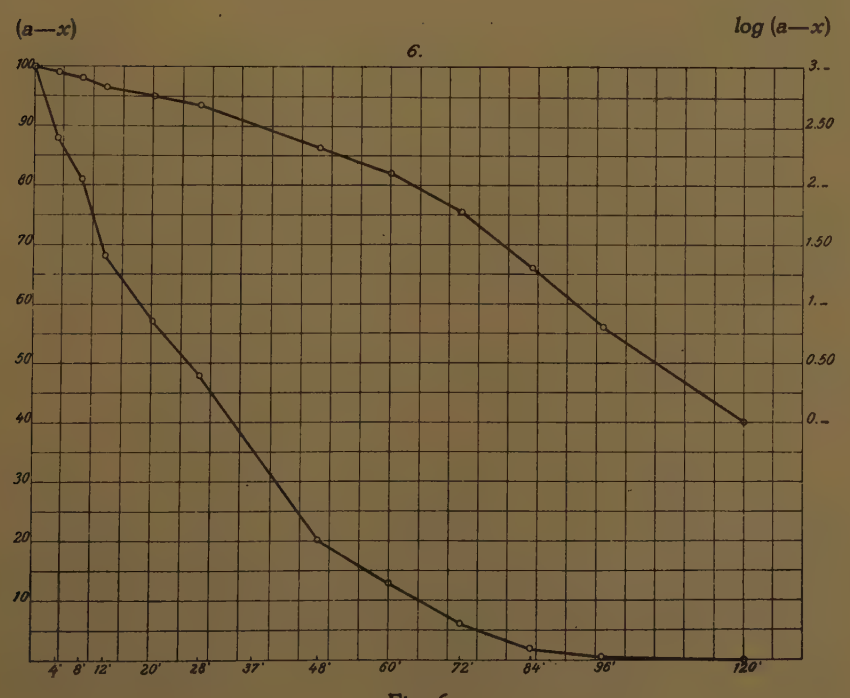

Fig. 6.

HAUDUROY [1]) concludes from his experiments on the influence of temperature on the bacteriophage that this organism is inactivated at tempe-

[1]) Arch. Intern. de Pharmac. dyn. et de Ther. 28. 1.

ratures of 90°—60° according as it is more or less virulent in the time of 5', whereas for all bacteriophages the mortal temperature is 102° during 5'. According to HAUDUROY this proves that besides the bacteriophage there must still be a substance secreted by the bacteria and transmitted to the liquid, which he believes to be answerable for the lysis. Now it is this very substance that is destroyed at a lower temperature, but the bacteriophage itself remains uninjured up to 102° for 5'. Here, then, we have to do with a complex.

WAGEMANS [1]) looked more closely at this subject, and gave a fine criticism on HAUDUROY's work. He arrives at the conclusion that the mortal temperature may differ largely for different bacteriophages but his results do not enable him to differentiate between inactivating-temperature and disinfecting-temperature.

As for the bacteriophage of Bacillus Danicus the following simple experiment shows its behaviour relative to the point under discussion.

A quantity of lysate was heated for 60 min. at 60°, after which the number of plages that rise to the surface on sowing 1 cc. of undiluted lysate, is so small that we can only just distinguish one or see none at all alternately. Nine plates were now inoculated in succession with 0.1 cc. and likewise 9 flasks filled with broth, made beforehand faintly turbid with bacteria. Next day the plates presented the following results.

4	plates	with	0	plages
3	„		1	„
1	„		2	„
1	„		3	„

pos. : neg. = 5 : 4.

The flasks were all turbid, but after some days the bacteria appeared to have been dissolved in some flasks.

4	flasks	clarified	
5	„	turbid	pos. : neg. = 4 : 5.

Filtration revealed that in the turbid flasks the bacteriophage could no more be reactivated; the clarified flasks gave a normal bacteriophage.

This, then, proves conclusively that in the experiments with Bacillus Danicus the bacteriophage disappears for good and all with the last plage.

It is quite comprehensible that initially all the flasks were turbid, with some slight individual variations, if we reflect that the four clarified flasks had been inoculated with one or some bacteriophages, whereas normally one drop of lysate, i.e. about 100.000 bacteriophages had been inoculated. The luxuriantly growing bacteria predominate absolutely at the outset, but because this bacteriophage can also dissolve older bacteria, a normal lysis

[1]) Arch. Intern. de Pharmac. dyn. et de Ther. 28. 176.

follows. This will not occur with a less virulent bacteriophage; the presence of the bacteriophage manifests itself only after filtration.

On the basis of this experiment we came to the same conclusion as WAGEMANS.

Already some years ago DE NECKER [1]) studied the disinfection of the bacteriophage quantitatively: he heated equal amounts of lysate for half an hour at temperatures rising from 40°—70° by leaps of 2 degrees. His conclusion is that the disinfection starts at 48°, and is accomplished after half an hour at 70°. As he employed the titrometrical method his values are only roughly approximate.

SCHUURMAN [2]) tries to substantiate his hypothesis that some of the bacteriophages in a definite lysate are provided with an envelop of ions, by the fact that in an unfavourable environment the bacteriophages first die off quickly and then slowly, as stated by APPELMANS [3]). In doing so SCHUURMAN makes a mistake, because exactly when all the bacteriophages are equivalent in a definite lysate, the disinfection proceeds logarithmically in the ideal case, i.e. first quickly, then slowly. Our observations go to show clearly that a division into two groups is out of the question.

CONCLUSIONS.

Disinfection of the bacteriophage of Bacillus Danicus does not reveal an individual variation curve. This does not go against its being a living organism, since living organisms, such as spores, also die off according to a logarithmic line.

A logarithmic disinfection-curve of this bacteriophage is fairly approximated with an acid or by radiation with ultraviolet light, but with an *increase* of the reaction-constant up to the double value towards the close of the experiment (2 % still alive); a comparatively larger *decrease* is found with thermal disinfection.

At 50° the reaction-constant is 0.0138, when only 0.04 % is still alive; at 60° it is 1.61 when 0.03 % is alive. We are tempted to roughly calculate from this the temperature-quotient:

$$Q_{10} = \frac{1.61}{0.0138} = 116.6.$$

We are not in a position to compare the incipient velocities as at 60° they cannot be measured, a large percentage being killed off in less than a minute. This calculation must therefore be looked upon as a rough estimate. So also this point does not clarify our knowledge of the nature of the bacteriophage; we include some temperature-quotients for the destruction of living cells and enzymes and for the denaturation of protein.

---

1) C. R. de la Soc. de Biol. 86. 736.
2) De bacteriophaag, een ultramicrobe. diss. Leiden 1925, pag. 29.
3) C. R. de la Soc. de Biol. 85. 1122.

Observer	Object	t. quotient
Collander [1])	plantcells	26 — 118
Chick-Martin [2])	Denatured protein	16 — 564
Lepeschkin [3])	id.	58 —9540
id. [4])	plantcells	6.2— 201
Zilva [5])	peroxydase	3000

The decrease of the reaction-velocity has also been observed by SENTER for the enzyme katalase [6]), and by EULER—LAURIN for saccharase [7]). A logarithmic disinfection for enzymes was detected i. a. by SÖHNGEN and SMITH [8]) for katalase (in the cell ; SENTER established it for katalase outside the cell) and TAMMAN for emulsin [9]).

After heating the bacteriophage during 50′ at 55° it appears that a lysate, derived from one of the surviving bacteriophages, dies off sooner than in normal cases. However, in the following generations a recovery reveals itself. This recovery can only be ascribed to life.

For the bacteriophage of Bacillus Danicus a so called inactivating temperature does not exist.

Now in view of the above results we are justified in stating that the shape of the disinfection-curve neither speaks for nor tells against its being a living organism, but that the gradual recurrence of a heated bacteriophage to its normal sensitivity lends support to D'HERELLE's theory which we consider quite admissible.

Laboratory for Microbiology, Wageningen
October 1925. *(Holland).*

[1]) Comment. Biol. Fennicae 1. 7—9. 1.
[2]) Journ. of Hygiene 10. 237.
[3]) Kolloid. Zeitschr. 31. 342.
[4]) Stud. Pl. Phys. Lab. Charl. Univ. Prague 1 5
[5]) Biochem. Journ. 8. (1914).
[6]) Zeitschr. Physik. Chem. 44. 257.
[7]) Hoppe—Seyler 108. 64.
[8]) Tijdschr. voor Vergel. Geneesk. 10. 2—3. 1.
[9]) Zeitschr. Phys. Chemie 18. 426.

Microbiology. — *"Determinations of Permeability with Saccharomyces cerevisiae"*.[1]) By N. L. SÖHNGEN and K. T. WIERINGA. (Communicated by Prof. G. GRIJNS.)

(Communicated at the meeting of October 31, 1925).

In determining, by the methods hitherto used, the amount of a substance that penetrates into the cell, and the progress of diffusion, a difficulty arises from the determination of the water-transport into and out of the cell. This is why a method has been looked for that would enable us to measure the water-transport precisely, and at the same time also the imbibition water in the cell-wall. A simple solution of this problem may be attained by adding to the liquid a substance that does not penetrate into the cell, that does not raise the osmotic pressure of the liquid to a degree worth mentioning, that can easily be determined quantitatively and is not adsorbed by the cell. By such a method we should be able to determine quantitatively how much water goes into or out of the cell, and how much of the dissolved substance (salts, sugar etc.) penetrates into the cell, while from a series of determinations, made at different intervals, the progress of diffusion could be deduced. From these values we could form an estimate of the amount of water, imbibed by the cell-wall.

We started our experiment by applying a large number of dyes, some of which were adsorbed, others were discoloured, so that a colorimetrical determination of the solution was impossible. Neither did experiments with soluble amylum yield practicable results, because amylum is adsorbed at the surface of the yeast-cells. Gelatin, however, appeared to satisfy the above-named conditions completely.

That this method admits of accurate observations is shown by the following experiments with lactose, resp. sodium chlorid in a 0.5 % gelatin-solution.

100 grms of yeast were suspended in 200 cc of increasing concentrations of a lactose solution in 0.5 % gelatin, and directly centrifuged off. Of the lactose (according to SCHOORL) and of the gelatin (according to KJELDAHL) the concentration was estimated in the liquid before and after the treatment with yeast. From the decrease in concentration of the lactose and of the gelatin we computed the amount of water that had to be added to 100 cc. of the original solution in order to obtain these decreases. (See table 1 : dilution in % water.)

Similar determinations with NaCl are tabulated in table 2.

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[1]) Pressed-yeast of the Ned. Gist- en Spiritusfabriek.

TABLE 1.

Concentration		m.g. lact. [1]) per 10 c.c. 1 % solution		Decrease	Dilution with % water	Titer of the gelatin		Decrease	Dilution with % water	Dilution lact. minus dilution gelatin
Gel.	Lact.	before	after			before	after			
0.5	0	0	0	—	—	23.8	23.8	0	0	—
„	1	99	91	8.0	8.8	23.8	23.8	0	0	8.8
„	2.5	98.75	90.25	8.5	8.8	23.4	23.3	0.1	0.4	8.4
„	5	99	89.5	8.5	8.5	23.1	23.1	0	0	8.5
„	10	100	89	11	12.4	22.1	21.4	0.6	3	9.4
„	15	100	87.5	12.5	14.3	21	20.2	0.8	4	10.1
„	20	99.5	86	13.5	15.7	20.1	19.2	0.9	4.7	11

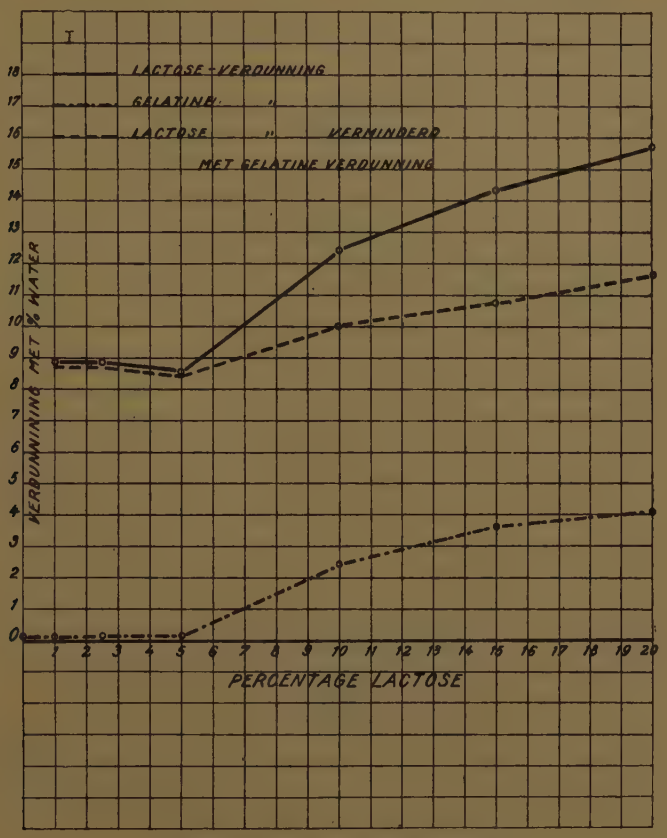
TABLE 2.

Concentration		c.c. $AgNO_3$ per 10 c.c. 1 % solution		Decrease	Dilution with % water	Titer of the gelatin		Decrease	Dilution with % water	Dilution NaCl minus dilution gelatin
Gel.	NaCl					before	after			
0.5	0	0	0	—	—	42.4	42.4	0	0	—
„	0.1	18	16.9	1.1	6	42	41.9	0.1	0.2	5.8
„	0.2	18	16.9	1.1	6	42.4	42.4	0	0	6
„	0.4	17.9	16.8	1.1	6.5	41.8	41.6	0.2	0.4	6.1
„	0.6	17.7	16.4	1.3	7.8	41.5	41.2	0.3	0.7	7.1
„	1	17.6	15.8	1.8	10.2	40.8	39.1	1.7	4.4	5.8
„	1.5	17.7	15.3	2.4	13.5	41.6	39.4	2.2	5.6	7.9
„	2	17.7	15	2.7	15.3	41.7	38.6	3.1	8.0	7.3
„	3	17.6	14.6	3	17.0	41.6	38	3.6	9.4	7.6
„	4	17.6	14.2	3.4	19.3	42.4	38	4.4	11.5	7.8
„	5	17.9	14	3.9	21.7	42.3	37.6	4.7	12.5	9.2

From these figures and the graphical representation I and II it appears, that with a concentration up to 5 % the lactose is diluted 8.7 % on an average. With concentrations of 10 % and upwards this amount increases considerably. The decreases of the concentration of the gelatin are smaller, but run quite parallel to those of the lactose. The wall of the yeast is impenetrable to the gelatin, while it admits lactose (imbibition water).

[1]) For the lactose-determination according to SCHOORL the original solutions were diluted to 1 % luctose.

Moreover the last column induces us to suppose that with 10 % lactose and upwards, plasmolysis has set in. However, this was not noticeable



Concentration lactose in perc.

——— Dilution of lactose.
—.—.— Dilution of gelatin.
— — — Dilution of lactose minus dilution of gelatin.

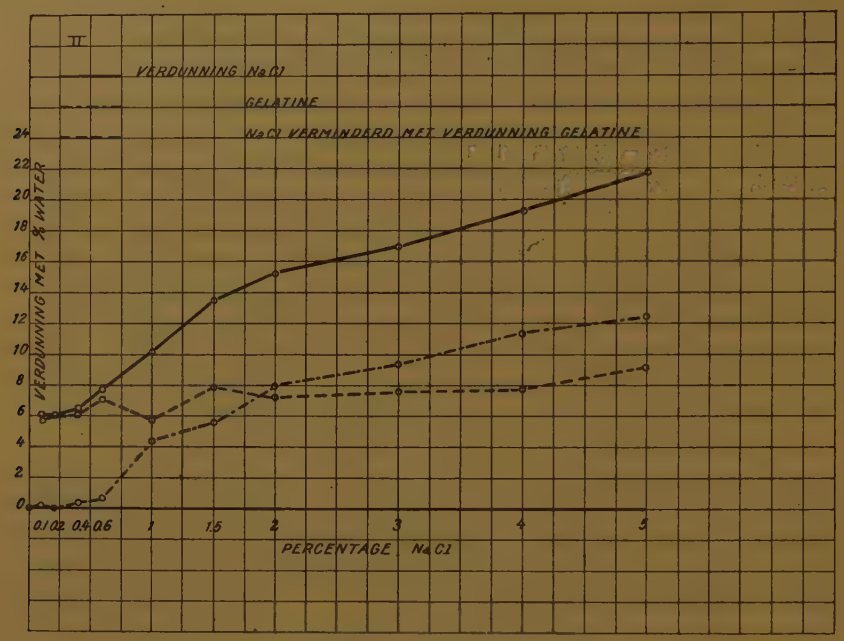
under the microscope, which is not surprising, since with the highest lactose-concentration only 5 % of the volume of the protoplasm retracts. That in NaCl solutions the water-transport from the cell occurs with much lower concentrations is of course due to a difference in osmotic power of the solution. It is quite possible to obtain by means of this method a reliable determination of isotonic coefficients.

Even with the highest NaCl concentrations no plasmolysis was noticeable under the microscope, although as much as 6 % of the space inside the cell wall must have become available, considering that the dilution caused by the imbibition water amounts to about 6 % and the highest total dilution is 9.2 %. However, the shape of the cells visibly altered.

It also appears from the figures that no adsorption has taken place.

Now, in order to ascertain whether substances penetrate into the cell

and at what rate this occurs, yeast suspensions of a definite concentration of the solution were exposed to roomtemperature, and at fixed intervals


VERDUNNING NaCl
GELATINE
NaCl VERMINDERD MET VERDUNNING GELATINE
VERDUNNING MET % WATER
0 2 4 6 8 10 12 14 16 18 20 22 24
0.1 0.2 0.4 0.6 1 1.5 2 3 4 5
PERCENTAGE NaCl

Concentration of NaCl in perc.

——— Dilution of NaCl.
—.—.— Dilution of gelatin.
— — — Dilution of NaCl minus dilution of gelatin.

the concentration of the dissolved substance and that of the gelatin was determined after part of the suspension had been deprived of the yeast-cells by centrifugalization. In this procedure it is expedient to treat yeast-suspensions in water in the same way and to deduct the values found by way of control. This also holds for the preceding and the following determinations.

Examples of determinations of this kind have been summarized in table 3 with NaCl, and in table 4 with ureum. In these determination 500 grm of yeast were suspended in 1 L. of a 1 % NaCl-, resp. 1 % ureum-solution in 0.5 % gelatin.

It now appears from table 3 that the rather considerable decrease of the concentration during the first 15 minutes, i.e. the time required for mixing up the yeast with the salt-gelatin solution, the titer of chlorin remains rather constant; after the first decrease the gelatin concentration remains unchanged. During the 48 hours' sojourn of the yeast in the 1 % NaCl solution no appreciable quantity of NaCl has diffused into the cell. The dilution computed from the lessening of the Cl-titer, directly after the mixing, amounted to 13.4 % water. This percentage rose in the course of 48 hours to 15 %, while the dilution of the gelatin averaged to about 5 %.

TABLE 3.

Diffusion 1 % NaCl

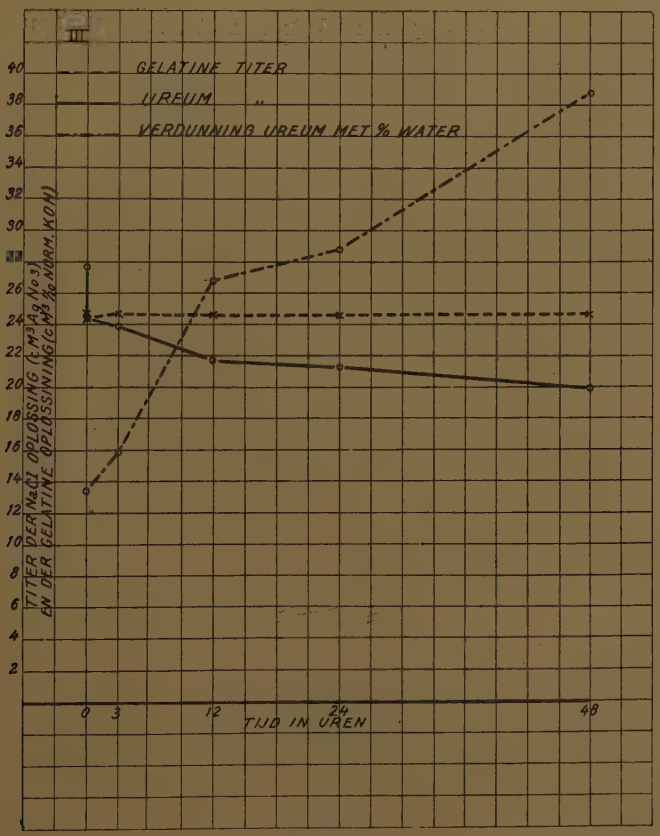
Time	Titer Cl	Titer gel.
0	16.2	23.9
15 min.	14.3	22.9
4 hrs.	14.25	22.5
12 „	14.2	22.6
24 „	14.2	22.9
48 „	14.1	22.6

TABLE 4.

Diffusion ureum

Time	Titer ureum	Dilution with % water	Titer gel.	Dilution with % water
0	27.7	—	24.7	—
15 min.	24.4	13.5	24.35	1.4
3 hrs.	23.9	15.9	24.75	—0.2
12 „	21.7	26.8	24.65	0.2
24 „	21.35	28.8	24.7	0
48 „	19.95	38.8	24.7	0

The latter amount is due to the water withdrawn from the cell, while 8.4 % of the decrease of the Cl-titer is brought about by imbibition water



Time in hours.

— — — Titer of gelatin.

——— Titer of ureum.

—.—.— Dilution of ureum with % of water.

in the cell-wall. Apparently the gelatin is come to a stop before the wall substance of the cell, while NaCl penetrates as far as the protoplasm.

Table 4 and the graphical representation III relate to the progress of diffusion of ureum, which was "Kjeldahled" in the gelatin-solution by precipitating the gelatin with Sodiumphosphotungstate. The ureum falls rapidly during the first 15 minutes (13.5 %) ; this continues regularly all the time of the experiment. Ultimately it rises to 38.8 %. The gelatin after a slight fall of 1.4 % at the beginning returns to the original concentration (passing plasmolysis).

This investigation, therefore, shows that ureum behaves quite differently from Sodium-chlorid and lactose — which, by the way, was also observed by other researchers — but it also proves that our method throws a clear light on quantity and the progress of diffusion.

Summary.

By the above method it is possible to determine quantitatively the penetration of compounds into the cell, and at the same time the progress of diffusion. Per gram of yeast, which has a superficies of about 1 m^2, about 0.25 mg. of NaCl, and 3.2 mg. of ureum diffuses in 43 hours. With ureum the rate of diffusion amounts during the first 24 hours to 2.2 mg. and during the last 24 hours to 1 mg. per gram of yeast.

At the same time the water-diffusion and the imbibition water in the cell-wall can be determined. With 1 % NaCl the amount of water withdrawn from the protoplasm was 8 % (table 2), the imbibition water 12 %. The imbibition water varies ; it amounts in table 1 to about 17 % of the yeast and in table 2 to more than 12 %.

Before long a detailed treatise will appear on this subject.

From the Laboratory for
Microbiology at the Agricultural University
of Wageningen.

October 1925.

Chemistry. — *"Equilibria in systems, in which phases, separated by a semi-permeable membrane."* XIV. — By F. A. H. SCHREINEMAKERS.

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Deduction of some properties of isotonic curves in ternary systems, in which dimixtion into two liquids occurs.

In order to deduce and to elucidate further the properties, discussed in the previous communication we contemplate fig. 1 in which $f_1\ q_1\ h_1$ and $f_2\ q_2\ h_2$ represent a part of the binodalcurve. The isotonic W-curve going through the two conjugated points is represented by the dotted curve $a\,b\,q_1\ c\,d\,q_2\,e$. We are able to represent the composition, the thermodynamical potential, etc. of an arbitrary phase Q, which contains the components W X and Y, not only with the aid of the quantities of those three components, but also with the aid of three arbitrary other phases (provided that they are not situated on a straight line); previously we have called those phases "composants" in distinction with the components, of which this phase Q consists [1]).

As many properties can be deduced more easily with the aid of composants than of components, we now shall use those composants.

We choose as composants 1. the diffusing substance W, 2. an arbitrary liquid b, 3. an arbitrary phase F. As we have seen formerly (l. c.) we can represent the composition of an arbitrary liquid p by:

$$m \text{ quantities of } W + n \text{ quantities of } F + (1-m-n) \text{ quantities of } b \quad (1)$$

so that we call b as fundamental composant. Consequently we have in fig. 1 a system of coordinates with the point b as origin and the lines $b\,W$ and $b\,F$ as axes. If we draw in the figure the lines $p\,p_1$ and $p\,p_2$ parallel to $b\,W$ and $b\,F$ then is (l. c.)

$$m = \frac{pp_1}{bW} \qquad n = \frac{pp_2}{bF} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we take for m as unity of length bW and for n as unity of length bF then we can put:

$$m = pp_1 \qquad n = pp_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In order to represent the composition of a liquid with the aid of com-

[1]) For a contemplation more in detail of components and composants comp.: In-, mono- and plurivariant equilibria. Comm. XXIV and XXV.

ponents in the communications XXIV and XXV we have used always x and y as variables, but, if we used composants, always m and n. As, however, now we do not compare both methods with one another and therefore no confusion can occur, we shall take for the composants also x and y as variables.

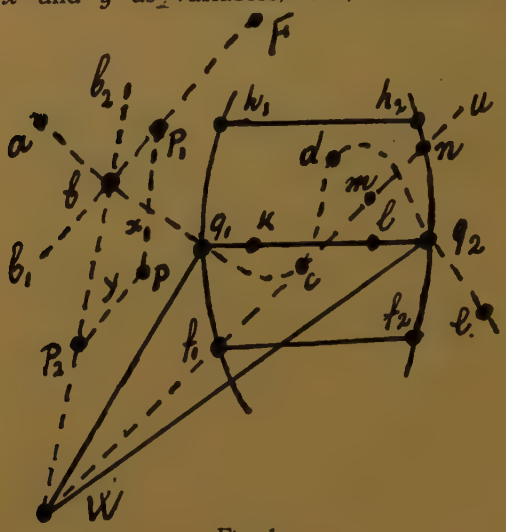

Fig. 1.

Consequently we represent the composition of an arbitrary liquid p by:

x quant. of $W + y$ quant. of $F + (1-x-y)$ quant. of b, (4)

so that in fig. 1 bW represents the X-axis and bF the Y-axis. Therefore we have in the figure:

$$x = pp_1 \qquad y = pp_2.$$

If we put in (4) $x = 0$ and $y = 0$ then p coincides with point b; for $y = 0$ p is situated anywhere on the line bW; for $y = 0$ and $x = 1$ p coincides with point W. If $x + y = 1$ then p is situated anywhere on the line WF. If we give a negative value to x or y or $1 - x - y$, then p falls outside the composants-triangle bWF.

We now take a liquid L with the composition as in (4) and a liquid L_1 with the composition:

x_1 quant. of $W + y_1$ quant. of $F + (1-x_1-y_1)$ quant. of b

and we now consider the osmotic equilibrium:

$$L \mid\mid L_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

in which the substance W only diffuses through the membrane. We assume that there are n quantities of L and n_1 quantities of L_1. If δn quantities of W (water) diffuse from L towards L_1, then x and y change with:

$$dx = \frac{nx - \delta n}{n - \delta n} - x = -\frac{(1-x) \cdot \delta n}{n - \delta n}$$

$$dy = \frac{ny}{n - \delta n} - y = \frac{y \cdot \delta n}{n - \delta n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

while x_1 and y_1 change with:

$$dx_1 = \frac{(1-x_1)\,\delta n}{n_1 + \delta n} \qquad dy_1 = -\frac{y_1\,\delta n}{n_1 + \delta n} \quad : \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The total thermodynamical potential of the osmotic system (5) now changes with:

$$(n-\delta n)\left(\zeta+\frac{\partial\zeta}{\partial x}dx+\frac{\partial\zeta}{\partial y}dy\right)+(n+\delta n)\left(\zeta_1+\frac{\partial\zeta_1}{\partial x_1}dx_1+\frac{\partial\zeta_1}{\partial y_1}dy_1\right)- \quad . \quad (8)$$
$$-n\zeta-n_1\zeta_1$$

in which ζ and ζ_1 represent the thermodynamical potentials of the liquids L and L_1. With the aid of (6) and (7), (8) passes into:

$$\left[\zeta_1+(1-x_1)\frac{\partial\zeta_1}{\partial x_1}-y_1\frac{\partial\zeta_1}{\partial y_1}-\zeta-(1-x)\frac{\partial\zeta}{\partial x}+y\frac{\partial\zeta}{\partial y}\right]\delta n \quad . \quad . \quad (9)$$

As the thermodynamical potential of a system in equilibrium is not allowed to change, (9) must be zero for infinitely small positive and negative values of ∂n. Consequently the osmotic system (5) is in equilibrium if:

$$\zeta+(1-x)\frac{\partial\zeta}{\partial x}-y\frac{\partial\zeta}{\partial y}=\left[\zeta+(1-x)\frac{\partial\zeta}{\partial x}-y\frac{\partial\zeta}{\partial y}\right]_1 \quad . \quad . \quad . \quad (10)$$

The *O. W. A.* (osmotic water attraction) of an arbitrary liquid is defined therefore by:

$$\varphi=\zeta+(1-x)\frac{\partial\zeta}{\partial x}-y\frac{\partial\zeta}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Previously (Comm. II) we have found, using components for the *O. W. A.*

$$\varphi=\zeta-x\frac{\partial\zeta}{\partial x}-y\frac{\partial\zeta}{\partial y};$$

the origin of the system of coordinates was situated then in point W and now in the point b.

We now replace the liquid L_1 of equilibrium (5) by the liquid b of fig. 1; we then have the osmotic equilibrium:

$$L \mid L_b \text{ (fig. 1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

As x_1 and y_1 for liquid b are zero, it follows from (10) that the liquid L is defined by:

$$\zeta+(1-x)\frac{\partial\zeta}{\partial x}-y\frac{\partial\zeta}{\partial y}=\left[\zeta+\frac{\partial\zeta}{\partial x}\right]_b \quad . \quad . \quad . \quad . \quad . \quad (13)$$

which represents the equation of the isotonic curve going through point b. From (13) follows:

$$[(1-x)\,r-ys]\,dx+[(1-x)\,s-yt]\,dy=0 \quad . \quad . \quad . \quad . \quad (14)$$

in which:

$$r=\frac{\partial^2\zeta}{\partial x^2} \qquad s=\frac{\partial^2\zeta}{\partial x\partial y} \qquad t=\frac{\partial^2\zeta}{\partial y^2}.$$

If we take the liquid L of equilibrium (12) in the vicinity of the point b (fig. 1) then x and y approach zero, r s and t rest finite, if b represents a ternary liquid. (14) now passes into:

$$r\,dx + s\,dy = 0 \qquad \frac{dy}{dx} = -\frac{r}{s} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

by which the direction of the isotonic curve in point b is defined. Of course this relation is valid for every arbitrary point of an isotonic curve, f. i. for the points a, q, c, d, q_2, e etc. but not for its terminating-points on the sides of the components-triangle.

As is known from the theory of the ternary liquids, outside the region of dimixtion is:

$$r > 0 \qquad t > 0 \qquad rt - s^2 > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

This is also the case on the binodalcurve itself. Within the region of dimixtion however a curve (not drawn in the figure) proceeds, on which:

$$rt - s^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This is the spinodalcurve, which is situated within the binodalcurve, but touches this in the critical points. Within this spinodalcurve is:

$$rt - s^2 < 0$$

and also may be $r \gtreqless o$ and $t \gtreqless o$. Is one of the magnitudes r or t negative, then $r\,t - s^2$ is negative also; of course the reverse is not the case.

If k and l in fig. 1 represent the points of intersection of the spinodal-curve with the line $q_1\,q_2$, then $r\,t - s^2$ is zero in those points, therefore; between q_1 and k and q_2 and l it is positive and between k and l negative.

The direction of the isotonic curve is defined by (15) in the point b; as b is situated outside the region of dimixtion, r is $> o$, but the sign of s is indefinite. If s is negative, then it follows from (15) that the isotonic curve is situated in the vicinity of the point b within the angle $W\,b\,F$ (and its opposite angle $b_1\,b\,b_2$); if s is positive, then the curve is situated within the angles $b_1\,b\,W$ and $b_2\,b\,F$. If $s = o$ then the curve touches in point b the Y-axis viz. the line $b\,F$. As r is never zero in the point b, the curve can, therefore, never touch the X-axis viz. the line $b\,W$ in b.

Above we have seen already that (15) is true for every arbitrary point of an isotonic curve; as in every point outside the region of dimixtion r is > 0, none of the lines Wa, Wq_1, Wq_2 and We can touch this curve, therefore. Hence follows the property, already discussed before:

the part of an isotonic W-curve, situated *outside* a region of dimixtion has such a form that every straight line, going through point W, intersects this curve in one point only and never touches it.

We now consider the part $q_1\,c\,d\,q_2$ of the isotonic curve, situated

within the region of dimixtion and we assume that r is zero in the points c and d and is negative, therefore, between c and d. It now follows from (15) that the isotonic curve touches the lines Wc and Wd in c and d. If we imagine within the angle cWd a straight line going through point W, then this intersects the isotonic curve in three points. Consequently we find:

the part of an isotonic W-curve, situated *within* a region of dimixtion *can* have such a form that we are able to draw from point W straight lines which touch this branch or intersect it in three points.

If r is positive in all points of the part of the isotonic curve, situated between q_1 and q_2, then for this part the same is true as for the part, which is situated outside the region of dimixtion. This is the case f. i. with the curves 4 and 6 of fig 1. (Comm. XIII).

In order to examine the binodal-curve in the vicinity of the points q_1 and q_2 we take as composants q_1 q_2 and W, we represent the composition of two arbitrary liquids L_1 and L_2 by:

x_1 quant. of $W + y_1$ quant. of $q_2 + (1-x_1-y_1)$ quant. of q_1

x_2 quant. of $W + y_2$ quant. of $q_2 + (1-x_2-y_2)$ quant. of q_1.

Consequently we take a system of coordinates with point q_1 as origin q_1 W as X-axis and q_1 q_2 as Y-axis. If L_1 and L_2 are two conjugated liquids, then the equilibrium $L_1 + L_2$ is defined by the three equations:

$$\left.\begin{aligned} \left(\zeta - x\frac{\partial\zeta}{\partial x} - y\frac{\partial\zeta}{\partial y}\right)_1 &= \left(\zeta - x\frac{\partial\zeta}{\partial x} - y\frac{\partial\zeta}{\partial y}\right)_2 \\ \left(\frac{\partial\zeta}{\partial x}\right)_1 = \left(\frac{\partial\zeta}{\partial x}\right)_2 \qquad & \left(\frac{\partial\zeta}{\partial y}\right)_1 = \left(\frac{\partial\zeta}{\partial y}\right)_2 \end{aligned}\right\} \quad . \quad . \quad , \quad (18)$$

We find those equations by expressing that the total thermodynamical potential of the equilibrium $L_1 + L_2$ does not change, if small quantities of each of the three components q_1 q_2 and W pass from the one liquid into the other. It follows from (18):

$$(x\,r + y\,s)_1\,dx_1 + (x\,s + y\,t)_1\,dy_1 = (x\,r + y\,s)_2\,dx_2 + (x\,s + y\,t)_2\,dy_2 \quad (19)$$

$$r_1\,dx_1 + s_1\,dy_1 = r_2\,dx_2 + s_2\,dy_2 \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$s_1\,dx_1 + t_1\,dy_1 = s_2\,dx_2 + t_2\,dy_2 \quad . \quad . \quad . \quad . \quad . \quad (21)$$

We now let coincide the liquids L_1 and L_2 with the points q_1 and q_2; therefore we have to put:

$$x_1 = 0 \quad y_1 = 0 \quad x_2 = 0 \quad y_2 = 1 \quad . \quad . \quad . \quad . \quad (22)$$

If we substitute those values in equations (19) and if we neglect the terms of higher order than the first, we find:

$$0 = s_2\,dx_2 + t_2\,dy_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The binodal-curve in the vicinity of the points q_1 and q_2 is defined, therefore, by (20), (21) and (23). Instead of (21) we now may write also:

$$s_1\,dx_1 + t_1\,dy_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

If we substitute in (20) the values of dy_1 and dy_2 which follow from (24) and (23) then we find:

$$\frac{r_1 t_1 - s_1^2}{t_1} dx_1 = \frac{r_2 t_2 - s_2^2}{t_2} dx_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Hence is apparent that dx_1 and dx_2 have always the same sign. This means: if a liquid is situated on $q_1 h_1$ ($q_1 f_1$) then the conjugated liquid is situated on $q_2 h_2$ ($q_2 f_2$).

Equation (24) defines the direction of the binodal curve in the point q_1; we have viz.:

$$\frac{dy_1}{dx_1} = -\frac{s_1}{t_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The direction of the isotonic curve is defined in every point by (15), in the point q_1 therefore, we have to give to r and s in (15) the values r_1 and s_1. We then have:

$$\frac{dy}{dx} = -\frac{r_1}{s_1} \qquad \frac{dy_1}{dx_1} = -\frac{s_1}{t_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

the first of which defines the direction of the isotonic curve, the second defines the direction of the binodal curve in the point q_1. As r_1 and t_1 are positive, $r_1 : s_1$ and $s_1 ; t_1$ have always the same sign, therefore. If $s_1 = o$ then follows:

$$\frac{dy}{dx} = \infty \qquad \frac{dy_1}{dx_1} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

It now follows from (29) and (28):

the binodal curve and an isotonic curve are situated in the vicinity of their point of intersection either both within the conjugation-angle or both within the supplement-angle. If the binodal curve touches the one leg of the angle, then the isotonic curve touches the other leg.

The *O. W. A.* of an arbitrary liquid L is defined by (11). For a liquid in the vicinity of L is true, therefore:

$$d\varphi = [(1 - x) r - y s] dx + [(1 - x) s - y t] dy \quad . \quad . \quad . \quad (29)$$

If we take the liquid L in the point q_1 (fig. 1) and if we take again the same components as above, consequently q_1 as origin of the system of coordinates, then x and y become zero. (29) then passes into:

$$d\varphi = r_1 dx + s_1 dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

If we proceed from q_1 along the binodal curve towards a point in the immediate vicinity, then the relation (24) is true for dx and dy. Hence follows for (30):

$$d\varphi = \frac{r_1 t_1 - s_1^2}{t_1} . dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

in which the coefficient of dx is positive. We now proceed from q_1 in the direction towards h_1 or, as we have expressed it in the previous communication: we proceed starting from the point q_1 along the binodal curve away from point W. As then dx is negative, $d\varphi$, therefore, is also negative and consequently the $O.\ W.\ A.$ increases. Therefore we find:

the $O.\ W.\ A.$ of the liquids of a binodal curve increases in that direction in which we move away from the point W.

We have already applied this property in order to define the direction in which the $O.\ W.\ A.$ of the liquids increases along the binodal curve of the figs. 1—3 (previous Communication).

In the previous Communication we have discussed already, that the isotonic curve, which goes through m_1 (figs 2 and 3 Comm. XIII), touches the binodal curve in this point m_1. A second branch of the isotonic curve, which is situated, however, totally within the region of dimixtion, also touches the binodal curve in the point m_2.

In order to examine the isotonic curve and the binodal curve in the vicinity of those points, we take as composants: W m_1 and an arbitrary phase F. Consequently we take a system of coordinates with m_1 as origin, m_1 W as X-axis and m_1 F as Y-axis.

For the isotonic curve, going through point m_1, equation (15) is true, in which we have to give to r and s the values, which they have in m_1. If we take those r_1 and s_1, then curve 2 (fig. 2 XIII) and curve 5 (fig. 3 XIII) in the vicinity of m_1 are defined by:

$$\frac{dy}{dx} = -\frac{r_1}{s_1} \quad \ldots\ldots\ldots\ldots \quad (32)$$

For an equilibrium $L_1 + L_2$ the equations (18) are true and the equations (19)—(21) which follow from this. We now imagine the liquids L_1 and L_2 in the points m_1 and m_2, so that:

$$x_1 = 0 \quad y_1 = 0 \quad y_2 = 0 \ldots\ldots\ldots \quad (33)$$

Limiting ourselves to terms of the first order, then (19)—(21) pass into:

$$0 = x_2 (r_2\, dx_2 + s_2\, dy_2)$$

$$r_1\, dx_1 + s_1\, dy_1 = r_2\, dx_2 + s_2\, dy_2$$

$$s_1\, dx_1 + t_1\, dy_1 = s_2\, dx_2 + t_2\, dy_2.$$

Hence follows:

$$r_1\, dx_1 + s_1\, dy_1 = 0 \quad \frac{dy_1}{dx_1} = -\frac{r_1}{s_1} \ldots\ldots \quad (34)$$

by which the direction of the binodal curve in m_1 is defined. It is apparent from (32) and (34) that the isotonic curve and the binodal curve touch one another in m_1.

If we take as composant m_2 instead of m_1, then we have to exchange

the indices 1 and 2 in the deduction above, hence follows that the isotonic curve and the binodal curve touch one another also in the point m_2 (figs. 2 and 3 XIII).

The change of the *O.W.A.* of a liquid L is defined by (29); therefore is true for the liquid m_1 (fig. 2 and 3 XIII):

$$d\varphi = [(1-x_1)\,r_1 - y_1\,s_1]\,dx_1 + [(1-x_1)\,s_1 - y_1\,t_1]\,dy_1 \quad . \quad . \quad (35)$$

As however $x_1 = 0$ and $y_1 = 0$, (35) passes into:

$$d\varphi = r_1\,dx_1 + s_1\,dy_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

We now choose the new liquid on the binodal curve so that dx_1 and dy_1 satisfy (34); then follows:

$$d\varphi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Hence follows the property, already formerly discussed:

the *O.W.A.* of the liquids of a binodal curve is maximum or minimum in the points, which are situated on the conjugation-line going through W (figs. 2 and 3 XIII).

We have assumed in the deductions above that the points m_1 and m_2 represent ternary liquids, so that $m_1\ m_2$ is a ternary conjugation-line.

If, however, m_1 and m_2 are binary liquids, then the deductions are valuable no more. If we imagine the line $Wm_1\,m_2$ (figs. 2 and 3 XIII) coinciding with one of the sides of the components-triangle, then t_1 and t_2 are infinitely large, but $y_1\,t_1$ and $y_2\,t_2$ rest finite for $y_1 = 0$ and $y_2 = 0$. It now follows from (19) — (21):

$$\left.\begin{aligned} y_1\,t_1\,dy_1 &= x_2\,(r_2\,dx_2 + s_2\,dy_2) \\ r_1\,dx_1 + s_1\,dy_1 &= r_2\,dx_2 + s_2\,dy_2 \\ t_1\,dy_1 &= t_2\,dy_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (38)$$

while (14) which defines the direction of the isotonic curve, passes into:

$$r_1\,dx + (s_1 - y_1\,t_1)\,dy = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

From (38) follows for the binodal curve

$$r_1\,dx_1 + \left(s_1 - \frac{y_1\,t_1}{x_2}\right)dy_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

It is apparent from (39) and (40) that the isotonic curve and the binodal curve do not touch one another now, a property to which we have pointed in the previous communication.

Above we have seen that the *O.W.A.* of the liquids of the binodal curve in the points m_1 and m_2 (figs. 2 en 3 XIII) is a maximum or minimum; we now shall consider this case more in detail.

We have represented the compositions of the liquids with the aid of the composants Wm_1 and F, in which F is an arbitrary phase. We now choose F in such a way that s_1 becomes $= 0$. (Later on it will

appear that F is situated then anywhere on the tangent going through point m_1). If we involve in (19) — (21) also terms of higher order and if we put:

$$x_1 = 0 \quad y_1 = 0 \quad y_2 = 0 \quad s_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

then we get:

$$\frac{1}{2} r_1 dx_1^2 + \frac{1}{2} t_1 dy_1^2 = x_2 (r_2 dx_2 + s_2 dy_2) + A_2 \quad . \quad . \quad . \quad (42)$$

$$r_1 dx_1 + \frac{1}{2} \frac{\partial s_1}{\partial y_1} dy_1^2 = r_2 dx_2 + s_2 dy_2 + B_2 \quad . \quad . \quad . \quad . \quad (43)$$

$$t_1 dy_1 + C_1 = s_2 dx_2 + t_2 dy_2 + C_2 \quad . \quad . \quad . \quad . \quad . \quad (44)$$

In the first part of (43) the terms with $dx_1\, dy_1$ and dx_1^2, which are infinitely small with respect to dx_1, are omitted. A, B and C contain the terms of the second order. We can satisfy those equations by taking dy_1 dx_2 and dy_2 of the same order and dx_1 of the order dy_1^2, while $r_2 dx_2 + s_2 dy_2$ is also of the order dy_1^2. Consequently we may write for (42)—(44):

$$\frac{1}{2} t_1 dy_1^2 = x_2 (r_2 dx_2 + s_2 dy_2) + A_2 \quad . \quad . \quad . \quad . \quad . \quad (45)$$

$$r_1 dx_1 + \frac{1}{2} \frac{\partial s_1}{\partial y_1} dy_1^2 = r_2 dx_2 + s_2 dy_2 + B_2 \quad . \quad . \quad . \quad . \quad (46)$$

$$t_1 dy_1 = s_2 dx_2 + t_2 dy_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Herein is:

$$A_2 = \frac{1}{2} \left(r + x \frac{\partial r}{\partial x} \right)_2 dx_2^2 + \left(s + x \frac{\partial r}{\partial y} \right)_2 dx_2\, dy_2 + \frac{1}{2} \left(t + x \frac{\partial s}{\partial y} \right) dy_2^2 \quad (48)$$

$$B_2 = \frac{1}{2} \frac{\partial r_2}{\partial x_2} dx_2^2 + \frac{\partial r_2}{\partial y_2} dx_2\, dy_2 + \frac{1}{2} \frac{\partial s_2}{\partial y_2} dy_2^2 \quad . \quad . \quad . \quad . \quad (49)$$

It follows from (45) and (46):

$$r_1 dx_1 + \frac{1}{2} \left(\frac{\partial s_1}{\partial y_1} - \frac{t_1}{x_1} \right) dy_1^2 = - \frac{1}{x_2} \left(\frac{1}{2} r\, dx^2 + s\, dx\, dy + t\, dy^2 \right)_2 . \quad (50)$$

It follows from (46) and (47):

$$t_1 s_2 dy_1 = - D\, dx_2 \quad t_1 r_2 dy_1 = D\, dy_2 . \quad . \quad . \quad . \quad . \quad (51)$$

in which:

$$D = r_2 t_2 - s_2^2 .$$

The terms of higher order are neglected in (51); if we substitute the values of dx_2 and dy_2 from (51) in the second part of (50), then we find:

$$r_1 dx_1 + \frac{1}{2} \left(\frac{\partial s_1}{\partial y_1} - \frac{t_1}{x_2} + \frac{t_1^2 r_2}{x_2 D} \right) dy_1^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (52)$$

by which the binodal curve is defined in the vicinity of the point m_1.

In a similar way we find from (14) for the isotonic curve:

$$r_1 dx + \frac{1}{2} \left(\frac{\partial s_1}{\partial y_1} - t_1 \right) dy^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

and for the change of the *O. W. A.* from (35):

$$d\varphi = r_1\, dx_1 + \frac{1}{2}\left(\frac{\partial s_1}{\partial y_1} - t_1\right) dy_1^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

We now choose dx_1 and dy_1 in such a way that the new liquid is situated on the binodal curve; consequently dx_1 and dy_1 must satisfy (52); then we may replace (54) by:

$$d\varphi = \frac{1}{2}\left(1 - x_2 - \frac{t_1 r_2}{D}\right)\frac{t_1}{x_2}\, dy_1^2 . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

Instead of by (37) $d\varphi$ is defined, therefore, by a magnitude of the second order. With the aid of (51) we are able to give still another form to (55), viz.:

$$d\varphi = \frac{1}{2}\left(1 - x_2 - \frac{dy_2}{dy_1}\right)\frac{t_1}{x_2}\, dy_1^2 . \quad . \quad . \quad . \quad . \quad (56)$$

It follows from (52) and (53) that the binodal curve and the isotonic curve are parabolic in the vicinity of m_1 and touch both the Y-axis in m_1. In order to define the position of those curves with respect to one another, we imagine in the figures to be drawn a line $m_1' W_1'$, parallel to and in the vicinity of $m_1 W$. For the point of intersection of $m_1' W_1'$ with those curves then is valid $dy = dy_1$. It follows then from (52) and (53):

$$\left(\frac{\partial s_1}{\partial y_1} - t_1\right) dx_1 = \left(\frac{\partial s_1}{\partial y_1} - \frac{t_1}{x_2} + \frac{t_1 r_2}{x_2 D}\right) dx \quad . \quad . \quad . \quad . \quad (57)$$

If we put:

$$\frac{\partial s_1}{\partial y_1} - t_1 = -\,Q_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

then we may write (57) with the aid of (55):

$$dx_1 - dx = 2\,\frac{d\varphi}{dy_1^2}\cdot\frac{dx}{Q_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

If we consider the value of Q_1 from (58), then follows from (53) that dx and Q_1 have the same sign, so that $dx : Q_1$ is always positive; the sign of (59) is the same, therefore, as that of $d\varphi$.

In order to apply the above equations, we shall distinguish different cases:

A. Binodal curve and isotonic curve in m_1; fig. 2 XIII.

The origin of the system of coordinates is situated, therefore, in point m_1 of the figure. If we imagine the conjugation-line $a_1 a_2$ in the vicinity of $m_1 m_2$, then we find:

$$\frac{a_1 m_1}{W m_1} > \frac{a_2 m_2}{W m_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

As $a_1 m_1 = d y_1$, $a_2 m_2 = d y_2$, $W m_1 = 1$ and $W m_2 = 1 - x_2$ in

which x is negative, therefore, we may write for (60), if we take positive dy_1 and dy_2, also

$$dy_1 > \frac{dy_2}{1-x_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

or:

$$1 - x_2 - \frac{dy_2}{dy_1} > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

If we take dy_1 and dy_2 both negative, then we find also (62).

As x_2 is negative, it follows from (56):

$$d\varphi < 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

Consequently φ is a maximum in m_1; the *O. W. A.* is a minimum in m_1, therefore. This is in accordance with the direction of the arrows on the binodal curve (fig. 2 XIII). It now follows from (59) in connection with (63):

$$dx_1 < dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

This means: if we proceed along the line $m_1'W_1'$ (see above) in the direction towards the point W, then we meet firstly the binodal curve and afterwards the isotonic curve; we see that this is in accordance with the figure.

B. Binodal curve and isotonic curve in m_1; fig. 3 XIII.

The origin of the system of coordinates is situated, therefore, in point m_1 of the figure; x_2 is positive now. In the same way as in *A* we find again (62); as, however, x_2 is positive, it now follows:

$$d\varphi > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

In accordance with the direction of the arrows in the figure, it follows, therefore, that the *O. W. A.* in m_1 is a maximum. In connection with (65) it follows from (59):

$$dx_1 > dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

This is in accordance with the position of the binodal curve and the isotonic curve in the vicinity of point m_1.

In order to consider the curves in the vicinity of the point m_2, we may use also the equations (52)—(59); then, however, we have to replace the index 1 by 2 and x_2 by x_1. We call those new equations (52^a)—(59^a); with the aid of (60) we find instead of (62):

$$1 - x_1 - \frac{dy_1}{dy_2} < 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

We now distinguish two cases.

C. Binodal curve and isotonic curve in m_2; fig. 2 XIII.

The origin of the system of coordinates is situated now in the point m_2 of the figure; x_1 is positive but smaller than 1. With the aid of (67)

we find from (56[a]) that $d\varphi < 0$, which is in accordance with (63), as is necessary.

Instead of (64) we find from (59[a]):

$$dx_2 < dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

This is in accordance with the position of the two curves in the vicinity of point m_2; the branch of the isotonic curve which touches the binodal curve in m_2 is situated viz. within the region of dimixtion.

D. Binodal curve and isotonic curve in m_2; fig. 3 XIII.

The origin of the system of coordinates is situated now in the point m_2 of the figure; x_1 is negative. With the aid of (67) we now find from (56[a]) that $d\varphi > 0$. This is in accordance with (65). Instead of (66) now is:

$$dx_2 > dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

This is also in accordance with the figure; the branch of the isotonic curve, which touches the binodal curve in m_2, is situated viz. within he region of dimixtion.

(*To be continued*).

Biochemistry. — *"The Spreading of Oxy-hemoglobin."* By E. GORTER and F. GRENDEL. (Communicated by Prof. P. EHRENFEST.)

(Communicated at the meeting of December 19, 1925.)

With the LANGMUIR [1])—ADAM [2]) apparatus we have examined the spreading of oxy-hemoglobin solutions, in the same way as these investigators have studied the spreading of simpler substances as fatty acids.

The oxy-hemoglobin was prepared according to HEIDELBERGER [3]). We started from horse's blood supplied by the abattoir, which was centrifuged. The red blood cells were then washed with chilled 0.85 percent sodium chloride solution. The blood corpuscles are then rinsed into a flask with a few cubic centimeters of water. To this toluene was added in amount equal to $^1/_7$ of the volume of the blood corpuscles used, cooled down to 0° C., and a mixture of 4 parts of carbon dioxide and 1 part of oxygen was introduced till a paste was obtained. This was continued for a few minutes with vigorous stirring, the flask was stoppered, and allowed to stand for 24—48 hours in the ice box. Then the oxy-hemoglobin has been crystallized out for the greater part. By centrifuging at low temperatures the crystals are separated from the thick paste. The crystal mass is drained in the ice box on a chilled porous plate, in which care is taken that desiccation at the surface is prevented by constantly renewing the surface layer. During this process a slow stream of carbon dioxide is directed over the surface of the plate. After this the oxy-hemoglobin is scraped into a chilled mortar, and ground to a smooth paste with sufficient ice-cold water to bring the final volume up to 3 to 3.5 times the weight in grams of the oxy-hemoglobin present in the original blood. Then this liquid is titrated with normal sodium carbonate solution to minimum turbidity, the whole mass again centrifuged in chilled tubes, and a stream of the carbon dioxide oxygen mixture passed into the chilled oxy-hemoglobin solution, until crystallisation begins. After having been kept in the ice-box for 24 to 48 hours, the oxy-hemoglobin is again crystallised out for the greater part, and a renewed re-crystallisation can take place in the same way as described above. The oxy-hemoglobin used by us was 5 $\times$ recrystallised in this way, and was used in a 12.5 % solution.

Experiments with fatty acids dissolved in alcohol or acetone had shown that almost equally good results could be obtained in the case of liquid fatty acids as with ether solutions, if only care is taken to bring the alcohol or acetone solution very carefully at the surface of the water in a small quantity. The result was, however, worse in the case of solid fatty acids: too small values were often obtained. Also the results obtained with

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[1]) LANGMUIR, I., J. Am. Chem. Soc., 1917, XXXIX, 1848.

[2]) ADAM, N. K., Proc. Roy. Soc. London, Series A, 1921, XCIX, 336; 1922, CI, 452, 516,

[3]) HEIDELBERGER, J. Biol. Chem., 1922, LIII 31.

emulsions of fatty acid in water, were much too low. Very remarkably the spreading from alcohol was often ½, from an emulsion often 1/3 of the spreading from ether.

When we now transferred the oxy-hemoglobin solution very carefully in a very small quantity on the water surface, we obtained at first very divergent results. We soon found that *time* had an important influence on the size of the surface occupied by the protein. This very clearly appears from the following curves (fig. 1 and 2).

When the protein is allowed to spread on water of a ph 7, a considerable time elapses before the spreading has reached its final value. Finally a constant value is found [1]), from which assuming a molecular weight of 16000, it can be calculated that the surface occupied by 1 molecule of oxy-hemoglobin amounts to :

$$1400—1600 \times 10^{-16} \text{ cm}^2.$$

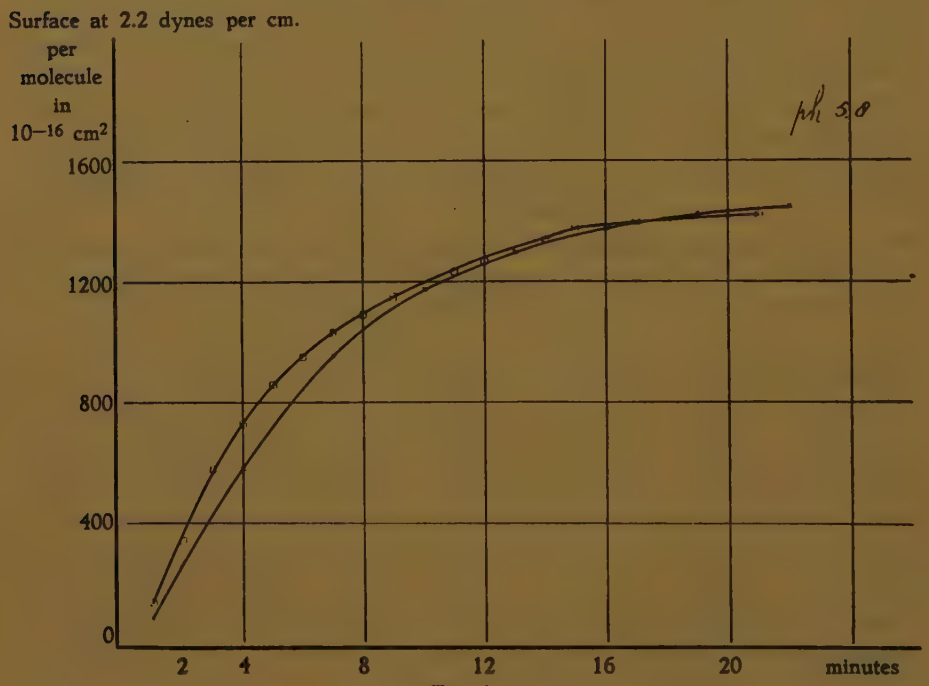

Fig. 1.

It now appeared that the time required for the spreading became very much smaller, when it was examined on a buffer mixture which reacted fairly acid. Thus the spreading was already almost immediately completed on a solution of 6.8 % acid potassium phthalate (ph 4), the experiment immediately giving the final result, when the ph of this buffer was reduced to 3 by the addition of HCl. (See curve 3.)

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[1]) Where not stated otherwise, the size was determined at a pressure of 2.2 dynes per cm. (50 mgr.).

That not the ph of the liquid alone determined this velocity, appeared when we examined the spreading on hydrochloric acid solutions of different

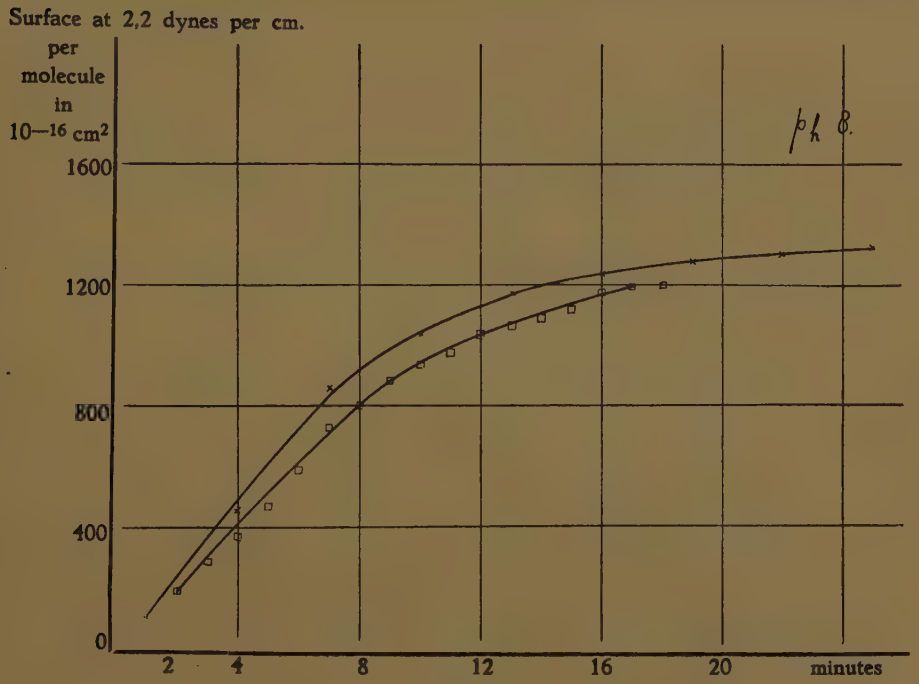


Fig. 2.

strengths. Only on a $^1/_{10}$ N. HCl-solution was the final value immediately reached, while even on a $^1/_{1000}$ N. HCl-solution with a time of observation of ½ hour hardly any increase of the initial spreading appeared to take place. (See fig. 3.) Probably the ph in the surface is the decisive factor, and this is much nearer the neutral point in the hydrochloric acid solution without buffer, because the protein and the carbonate in the hemoglobin solution bind hydrochloric acid.

This phenomenon could not be followed towards the alcaline side, because the blank values become too great there. Then the water is too sensitive to capillary active contaminations. On a phosphate mixture of a ph 8 the spreading was slow. (Curve 2).

After we had observed this gradual spreading and had interpretated it as the disintegration of an associated layer of molecules, an article by CARY and RIDEAL [1]) came under our notice, in which the same phenomenon is described in fatty acids. If these are placed as such on a spot of the water surface, the spreading takes place according to a line similar to that observed by us for proteins.

Moreover the rapidity of the spreading of hemoglobin appeared to be dependent on the *temperature*. Thus on neutral water the rapidity of the

[1]) CARY und RIDEAL, Proc. Roy. Soc. London, Series A, October 1925.

spreading was increased by a rise of temperature, so that at 45° C. the final spreading is already reached after a few minutes.

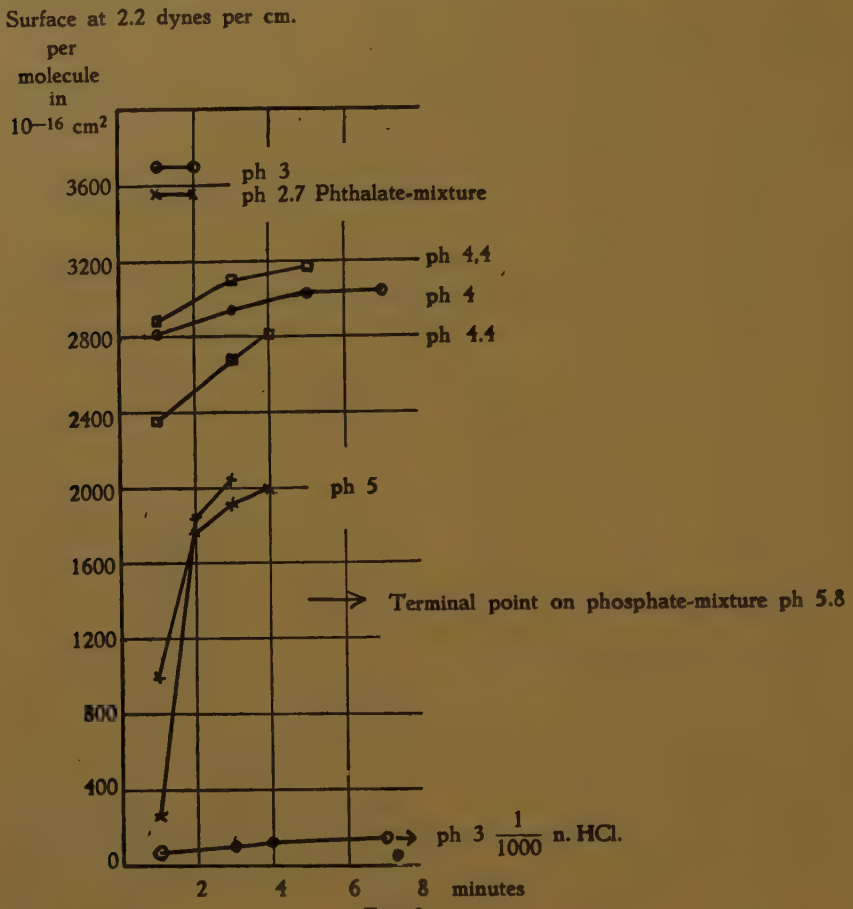


Fig. 3.

But not only the rapidity of the spreading appeared to vary, also the surface occupied by the final spreading by the hemoglobin was not always of the same size. On this, too, ph and temperature had a very marked influence. The greatest values were obtained on buffer mixtures with a ph < 3 when the experiment was made at a temperature of 15° C., the maximum spreading being already reached on buffer mixtures with greater ph at higher temperature. If for this maximum spreading the surface per molecule is calculated (assuming again 16000 for the mol. weight), 3200×10^{-16} c.m². is found. Very remarkably this is double the value found in the neighbourhood of the neutral point (ph 5.5—8). The non-buffered hydrochloric acid solutions exhibited the same phenomenon also in their influence on the extent of the final spreading: on a $^1/_{10}$ N. hydrochloric acid solution maximum spreading (3200×10^{-16} c.m².), on a $^1/_{100}$ N. hydrochloric acid solution half the maximum spreading, and on a $^1/_{1000}$ N. hydrochloric acid solution often scarcely more than

100×10^{-16} c.m^2. These results, too, were a function of temperature. The above data are valid for a temperature of 15° C.; higher temperature promotes the maximum spreading. The influence of the ph appears from the following curve.

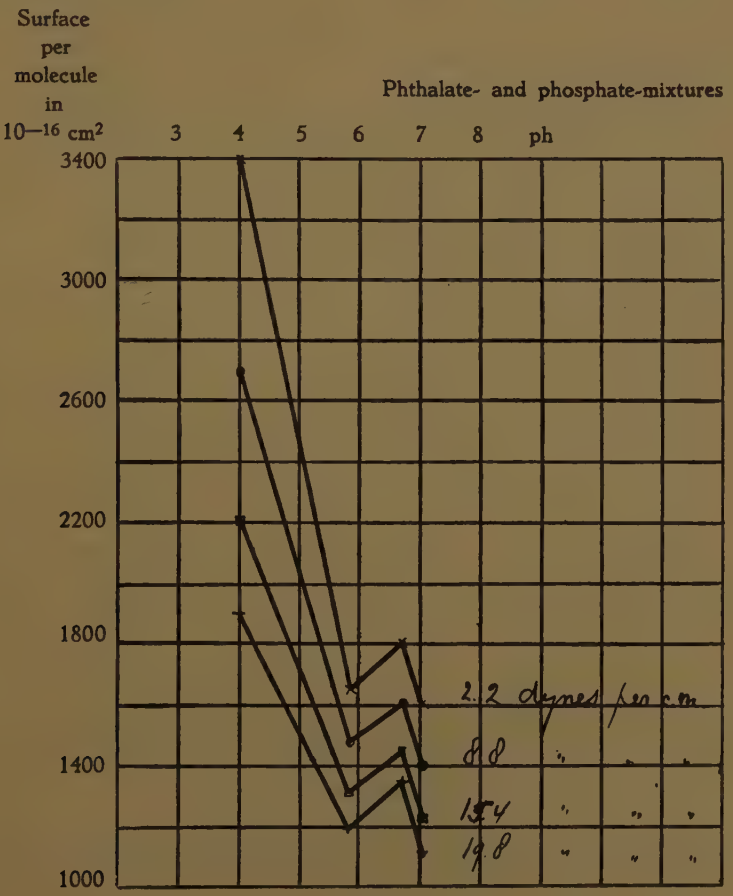


Fig. 4.

It is interesting to note that at the iso-electric point of the oxy-hemoglobin a greater value is constantly found on buffer mixtures than on mixtures with somewhat greater or smaller ph.

It is also noteworthy that the maximum final spreading, as it is reached on acid liquids, becomes little, if at all, greater by a rise of the temperature.

When the influence of the *pressure* on the protein film at the surface is studied, it appears to be greatly dependent both on temperature and on the ph of the water.

At laboratory temperature and on buffer mixtures of a ph between 5.8 and 7 the pressure surface curve is the same (Cf. curves 5, 6, and 7).

From a pressure from 4 tot 5 dynes per c.m. the line runs straight with the same slope. In comparison with the line of fatty acids (examined on acid) this slope is much less steep; i.e. the protein film has greater

compressibility. The force required to break the film lies at about 20—22 dynes. The absolute measure of the molecule being about the same in all

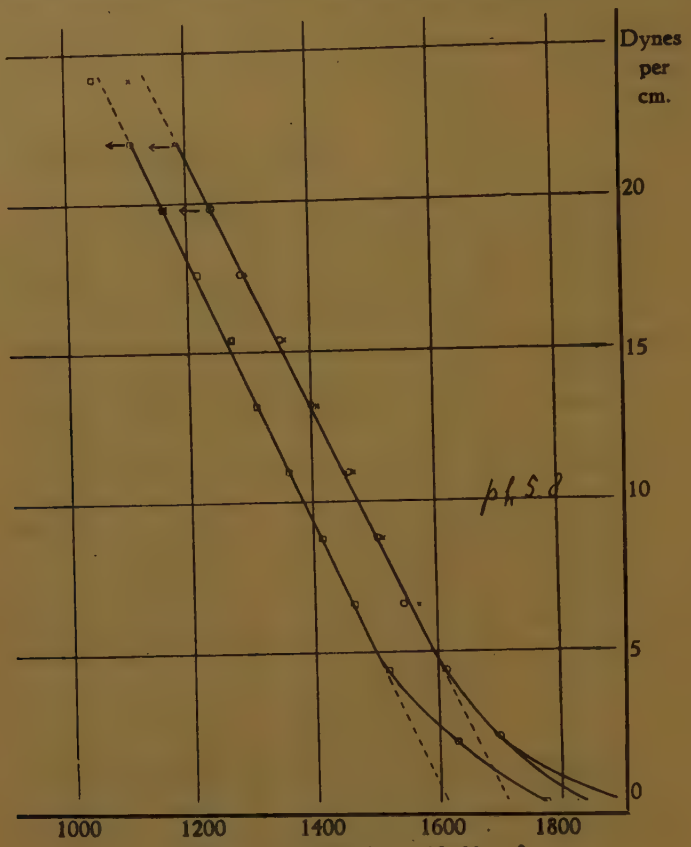


Fig. 5.

these experiments, it follows from the parallelism of the lines that also the force required to reduce the film to half its size has the same value (36 dynes per cm.); hence the constant of compressibility has also the same value.

At laboratory temperature and a ph under 3 we find an entirely different course of the pressure-surface curve. The point where the line begins to run straight, lies at higher pressure (6.6—8.8 dynes), and the slope of the straight portion of the line is greater, even if a correction is applied for the double size of the molecule by the choice of another scale. This may be thus expressed in a figure, that with greater hydrogen ion concentration the film can be diminished to half its size more easily, i.e. by a smaller force. (Here 24 dynes per cm.). The constant of compressibility becomes therefore greater with increasing ph.

Here too the same thing holds mutatis mutandis for HCl-solutions as was already stated for the rapidity and extent of the spreading. (Curve 8).

Here the same influence of the increase of the temperature may again

be observed. (Curves 9 and 10). While e.g. with a ph 5.8 and at a temperature of 15° C. 36 dynes are required for the reduction to half its

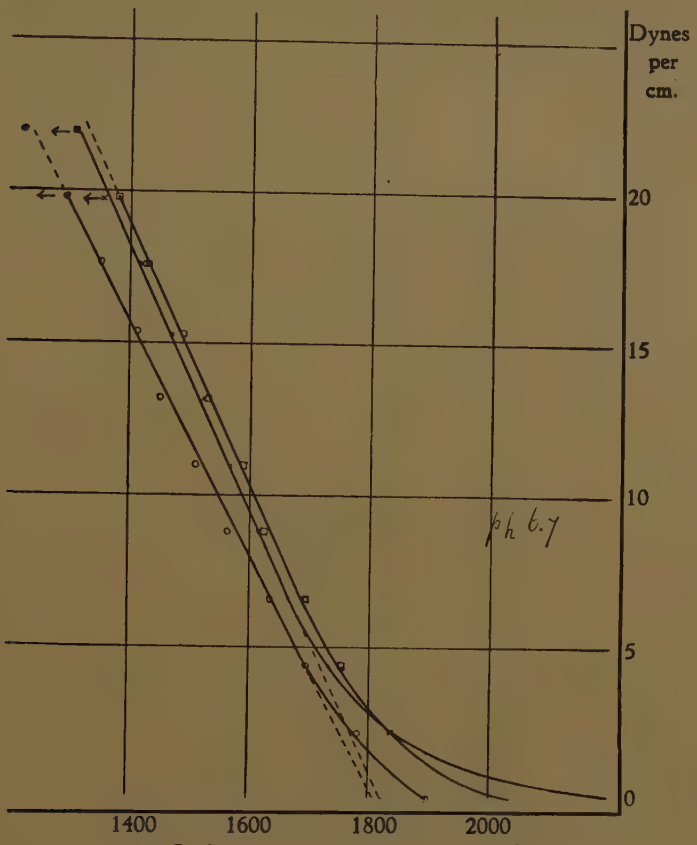


Fig. 6.

size (in the straight part of the line), 24 dynes are sufficient to obtain this result at 45° C.

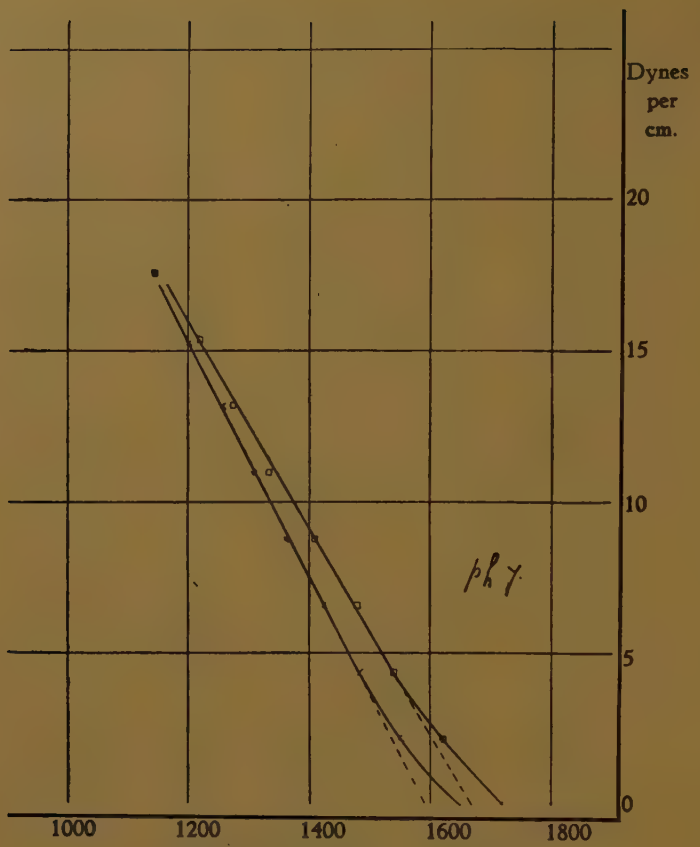
This result might have been expected, because also on the size of the spread film rise of temperature and increase of hydrogen ion concentration had the same influence.

By a suitable choice of a ph and temperature the two forms of compressibility may sometimes be seen to pass into each other in the same experiment (Cf. curve 11).

The influences of pressure and temperature on the extent of the spreading are *reversible*, unless the film has been broken.

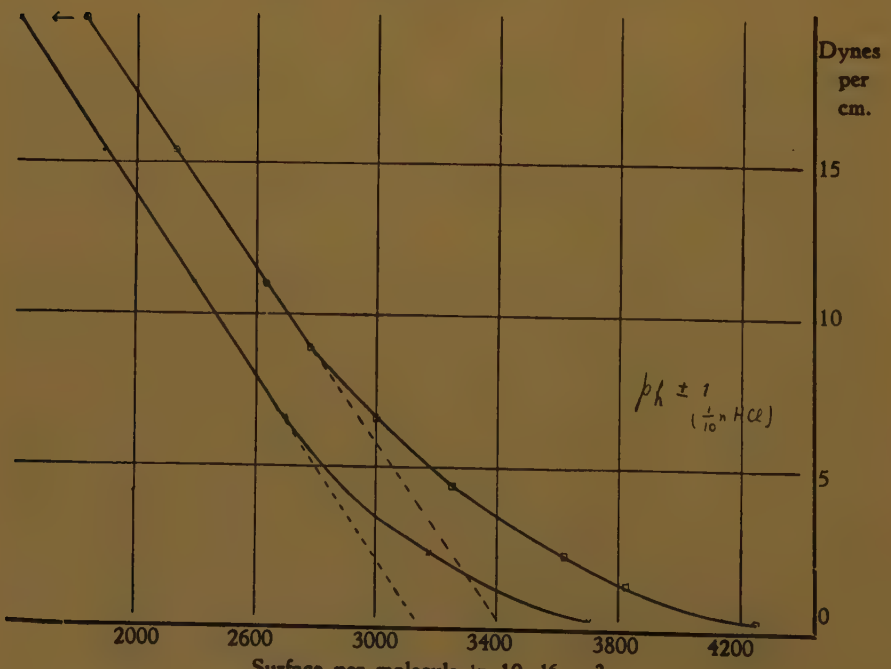
The first part of the line up to the point where it begins to run straight, has not yet been discussed. Already MARCELIN [1]) has pointed out that in fatty acids the corresponding line follows the law of BOYLE. Under small pressure the molecules at the surface behave as in a diluted solution or a gas. Also our measurements on proteins are in harmony with this view.

[1]) MARCELIN, Société de chimie physique, Décembre 1924.



Surface per molecule in 10^{-16} cm^2.

Fig. 7.



Surface per molecule in 10^{-16} cm^2.

Fig. 8.

When we now would try to give an explanation of the phenomena observed, we must first of all remember the structure of the proteins, in which amino acids are mutually bound, and CO—NH-groups are formed. These are polar with regard to water. It can be easily calculated [1]) that of the latter and of free NH_2-groups there are ± 180 per molecule present in oxy-hemoglobin. From this it may be calculated that starting from the greatest number found by us for the spreading, every CO—NH group occupies a space of about $\frac{3200}{180} = \pm 18 \times 10^{-16}$ cm². on the surface. When the extent of the spreading of a substance as a fatty acid or an amine is compared with this, which spreads 20×10^{-16} cm². due to the

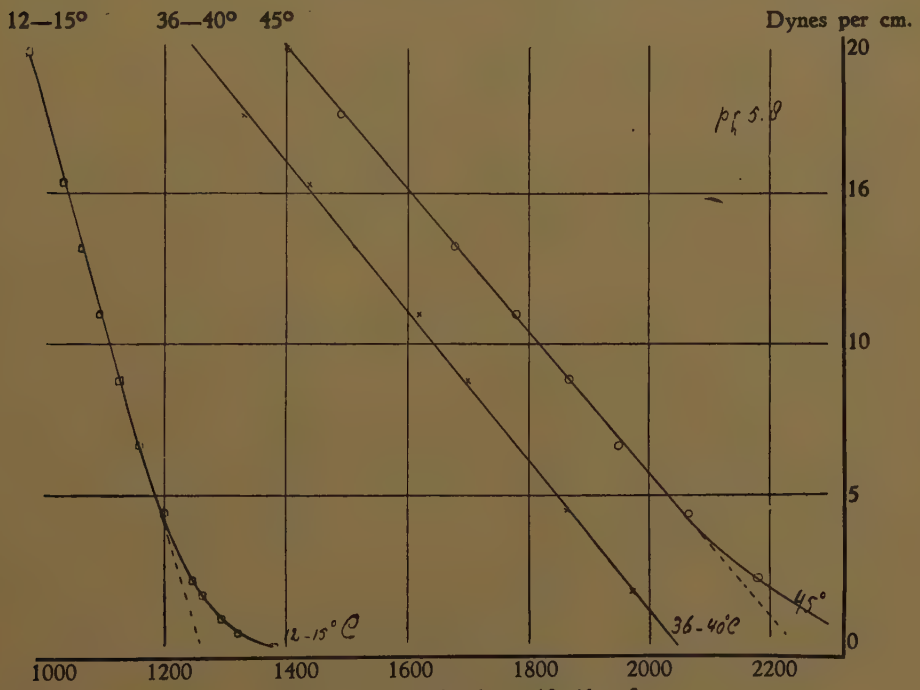


Fig. 9.

presence of a COOH or NH_2-group, the agreement is very good. The CO—NH-group can, accordingly, occupy no more than 20 A.U. in a spreading substance. There is, therefore, some reason to assume that the maximum velocity of spreading on the acid or heated water-surface of the oxy-hemoglobin is the real mono-molecular spreading (film a), depending on the circumstance that about all the CO—NH-groups have been drawn to the water-surface.

The values being half so great on neutral water (film b) it must be

[1]) From the values for the separate amino acid groups, as given by PLIMMER, Chemical constitution of the proteins, Part I Analysis, 1917, 132.

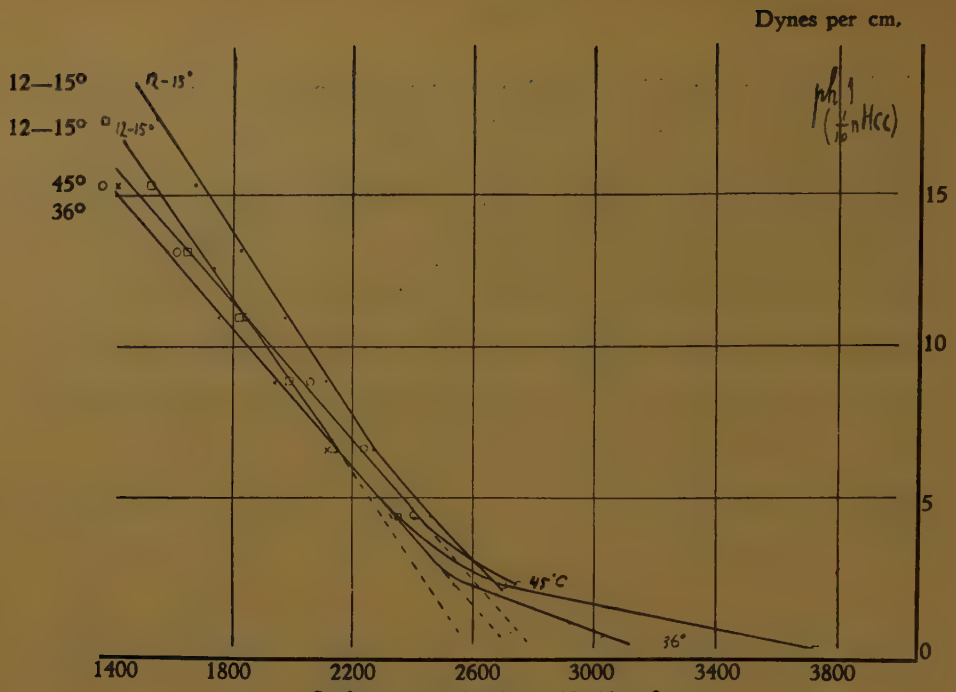

Dynes per cm.
12—15°
12—15°
45°
36°
12-15°
45°C
36°
15
10
5
0
1400
1800
2200
2600
3000
3400
3800
Surface per molecule in 10—16 cm².

Fig. 10.

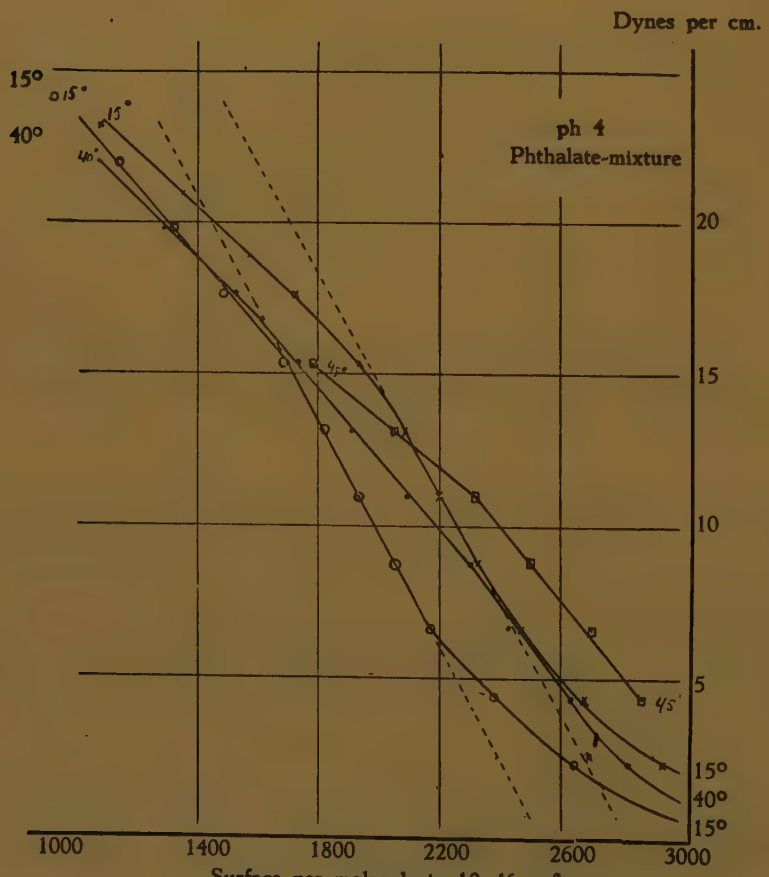

Dynes per cm.
15°
40°
ph 4
Phthalate-mixture
20
15
10
5
15°
40°
15°
1000
1400
1800
2200
2600
3000
Surface per molecule in 10—16 cm².

Fig. 11.

assumed that only half of the polar groups participate in the spreading, or that in this case always two molecules lie associated, the layer being bi-molecular under these circumstances.

The latter supposition seems more probable, because acid and rise of temperature both as a rule counteract association, and because from the low values found for the initial spreading a strong association of the oxy-hemoglobin may be inferred.

Besides in this connection we wish once more to point out that the maximum spreading (to ± 3200 A.U. per molecule) obtained on a buffer liquid of ph 3, can no more be increased by heating than inversely the maximum spreading obtained by heating, by acidification.

On the other hand the bi-molecular spreading (± 1600 A.U.) can be doubled both by heating and by acidification of the water.

Assuming that the spreading to 3200 A.U. per molecule is the mono-molecular spreading, we find that the thickness of the layer must be ± 6 A.U. For the molecular volume is 20000, when the Sp. Gr. is assumed to be 1.275. Though the smallest spreading (film c) can never be measured accurately, we wish yet to state that it repeatedly amounted to about $^1/_{50}$ of the maximum spreading, and that the size of the hematine, which very easily spreads from ether, is of the same order of magnitude, viz. ± 70×10^{-16} cm^2. Perhaps this gives some insight into the association of the molecules in the oxy-hemoglobin solution used.

Also in a somewhat different way the values found for the spreading of the protein might be accounted for: viz. by assuming that the protein molecule is a rectangular prism of the following dimensions 6, 12 and 280 A.U., in which polar groups are present in all three planes. The smallest plane $6 \times 12 = 72$ A.U^2. is then the initial spreading and the size of the hematine molecule; the largest plane 12×280 A.U^2. explains the final spreading, the bi-molecular spreading being 6×280 A.U^2. On the other assumption it must also be assumed that the thickness of the molecule is 6 A.U., but then the shape of small flat blocks with a surface of 3200 A.U^2. may be assigned to the molecule. If this plane is a square, 56.5 A.U. is found for length and breadth (fig. 12). When groups of 48 molecules associated are considered, this difference disappears, as appears clearly from figure 12.

Summary:

On a water-surface there exist three forms a, b and c of a hemoglobin film, which differ from each other in many respects. One film (a) is the maximum spreading (± 3200 A.U. per molecule), is more easily compressible, and is formed either by increase of the hydrogen ion concentration or by rise of temperature. The other film (b) has half the extension and is found at a hydrogen ion concentration between 5.5 and 8 and at lower temperature. There occurs a third film (a), which is very unstable, often undergoes little change only on a hydrochloric acid solution

of $^1/_{1000}$ normal, and exists a very short time at the beginning of the experiment also on neutral liquids.

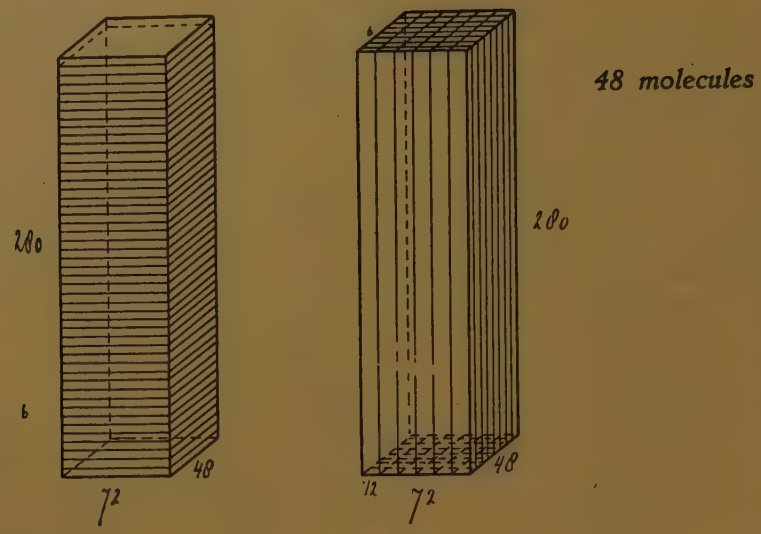


Fig. 12.

Application.

The supposition that the hemoglobin molecules are long needles, at the extremity of which the hematine particle is found with the active iron atom in the centre, is rather attractive. Thus all the iron atoms would be placed in one plane. If it is calculated how many of these groups of 48 molecules can find room in one red blood corpuscle in such a way that the iron atoms lie free on a surface, it appears that such a grouping may be reached by constructing spheres, at the surface of which the hemoglobin molecules lie with the iron towards the outside. In this way exactly half of all the hemoglobin may be got rid of. Hence it is possible to dispose of all the hemoglobin either by giving every sphere a double layer, or by imagining that on an average a double layer of these spheres has been placed along the surface. We mention the result of this calculation only because it is in accordance with the law of BÜRKER, viz. that the proportion of the quantity of hemoglobin and the surface of the chromocytes of different animal species is constant. In this connection we will also point out that it has been established by us that the lipoids of the whole chromocyte occupy a bi-molecular layer [1]) at the surface.

In conclusion it may be added that the doubling of the spreading of the hemoglobin under influence of acid, also appeared to be valid for other proteins, that the film of these proteins must have the same order of thickness, and that we consider it possible that a theory of muscular contraction should reckon with this influence of acid on the size of the protein molecule.

[1]) GORTER and GRENDEL, Journal of exp. medicine Apr. 1 1925 vol. XLI p. 439.

Physics. — *"The determination of the potentials in the general theory of relativity, with some remarks about the measurement of lengths and intervals of time and about the theories of* WEYL *and* EDDINGTON." By Prof. H. A. LORENTZ.

(Communicated at the meeting of March 24, 1923).

§ 1. It was remarked some years ago by KRETSCHMANN [1]) that from observations solely concerning the course of rays of light and the motion of material particles, the values of the potentials g_{ab} which characterize a gravitation-field can be so far deduced that only a constant factor remains undetermined. He showed, in fact, that, if two sets of values g_{ab} and $\overline{g}_{ab}$ are in agreement with these observations, the ratio $\overline{g}_{ab}/g_{ab}$ must be the same for all suffixes a and b, and independent of the coordinates.

It is easily seen in what manner this determination of the potentials can be effected. Let us imagine that a physicist explores a gravitation-field by attending to the motion of light-signals and material particles which he throws into the field in any way he likes, and let us suppose that he does so a great number of times and under varied conditions. His object will be to note and clearly to record the encounters between these projectiles.

To this effect he might tabulate the encounters in a register, after having numbered the projectiles, but a better picture of the phenomena may be obtained by means of a diagram drawn in a four-dimensional space R_4 in which each projectile has its „world-line". An encounter between two projectiles will be represented by an intersection of their world-lines and the lines have to be drawn in such a way that, along any one of them, the intersections with other lines follow each other in the order in which the successive encounters have taken place.

It is clear that the observer has a good deal of liberty in the construction of the diagram. A particular figure will continue to serve his purpose though it be subjected to an arbitrary deformation, provided only that the connexions between its parts remain unbroken. Even, as we are concerned with the intersections and their order only, all diagrams thus derivable from each other may, in a sense, be said to be the *same* figure. It ought also to be remarked that points in R_4 are defined exclusively by the intersection of world-lines, there being, according to the conceptions of the theory of relativity, no other means for defining the position of a point. Each point represents an „event".

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[1]) E. KRETSCHMANN, Ueber den physikalischen Sinn der Relativitätspostulate, Ann. d. Physik, (4) **53** (1917), p. 575.

We shall suppose the word-lines and the encounters to be so numerous that we may speak of a continuous succession of points along each line, and also of lines lying infinitely near each other.

After having drawn the world-diagram we can introduce coordinates, assigning to each point four numbers $x_1 \ldots . x_4$. In doing so we are limited only by the restriction of continuity and by the condition that, as we proceed along a world-line in the positive direction, corresponding to the succession of the encounters, the time-coordinate x_4 must constantly increase.

§ 2. EINSTEIN's theory postulates the possibility of associating with each point in R_4 ten numbers g_{ab} $(g_{ba} = g_{ba})$, such that, if one puts

$$ds^2 = \Sigma\,(ab)\, g_{ab}\, dx_a\, dx_b\,,$$

ds being the „line-element", the world-line of a flash of light satisfies the condition

$$ds = 0$$

and that the world-line of a material particle is a geodesic, i.e. such that, if its beginning and its end are kept fixed,

$$\delta \int ds = 0.$$

Admitting this, and remembering that the values of the coordinates can be directly read from the diagram, we may solve our problem as follows.

We consider in the first place the world-lines belonging to light-signals and passing through a definite point P. Let, on any one of these, Q be a point infinitely near P. We see at once the values of the four differentials dx_a, corresponding to the transition from P to Q, and the condition

$$\Sigma\,(ab)\, g_{ab}\, dx_a\, dx_b = 0$$

gives us a homogeneous linear relation between the potentials, with known coefficients $dx_a\, dx_b$. Proceeding in the same way with eight other lines of the same class, passing through P, we are led to nine equations from which the ratios between the potentials may be found. This can be done for any position of the point P and the result may be written in the form

$$g_{ab} = \omega\, \gamma_{ab}, \qquad (1)$$

where the quantities γ_{ab} are known functions of the coordinates $(\gamma_{ab} = \gamma_{ab})$, whereas the function ω remains to be determined. If, as we shall suppose, the field is free from discontinuities, the quantities γ_{ab} may be chosen as continuous functions, and then ω will be of the same kind.

Any world-line of light passing through P and not included in the group selected will give a verification of the theory, because for it also the equation

$$\Sigma\,(a\,b)\, g_{ab}\, dx_a\, dx_b = 0$$

must hold.

§ 3. The world-lines for particles, the geodesics, may now serve for the determination of the function ω. Indeed, at any point of such a line, we have the four equations

$$\frac{d^2x_c}{ds^2} = -\Sigma\,(ab)\begin{Bmatrix} a\,b \\ c \end{Bmatrix}\frac{dx_a}{ds}\frac{dx_b}{ds} \tag{2}$$

and these may be put in the form of differential equations for ω.

The symbol $\{\ \}$ in (2) is defined by

$$\begin{Bmatrix} a\,b \\ c \end{Bmatrix} = \Sigma\,(e)\,g^{ce}\begin{bmatrix} a\,b \\ e \end{bmatrix} = \tfrac{1}{2}\,\Sigma\,(e)\,g^{ce}\,(g_{ae,b} + g_{be,a} - g_{ab,e}),$$

$g_{ac,b}$ being the differential coefficient of g_{ac} with respect to x_b and the set of quantities g^{ab} the "inverse" to the set g_{ab}. Similarly, we may introduce γ^{ab} and the derivatives $\gamma_{ab,c}$. All these quantities will have definite values because γ_{ab} is known.

Now

$$g^{ab} = \frac{1}{\omega}\,\gamma^{ab}, \qquad g_{ae,b} = \frac{\partial}{\partial x_b}\,(\omega\,\gamma_{ae}),$$

and therefore

$$\begin{Bmatrix} a\,b \\ c \end{Bmatrix} = \tfrac{1}{2}\,\Sigma\,(e)\,\gamma^{ce}\,(\gamma_{ae,b} + \gamma_{be,a} - \gamma_{ab,e}) + \\ + \tfrac{1}{2}\,\Sigma\,(e)\,\gamma^{ce}\left(\gamma_{ae}\frac{\partial\,log\,\omega}{\partial x_b} + \gamma_{be}\frac{\partial\,log\,\omega}{\partial x_a} - \gamma_{ab}\frac{\partial\,log\,\omega}{\partial x_e}\right). \tag{3}$$

If, further, we put

$$d\sigma^2 = \Sigma\,(ab)\,\gamma_{ab}\,dx_a\,dx_b\,, \tag{4}$$

we have

$$ds = \sqrt{\omega}\,d\sigma. \tag{5}$$

The differential $d\sigma$ may be considered as the line-element expressed in a new measure, and since by (4) it is known, we know also for any point of the geodesic under consideration the value of the integral $\sigma = \int d\sigma$, reckoned from some fixed point of the line. Thus, along the line, the coordinates become known functions of σ, and the same may be said of their first and second derivatives with respect to that variable.

Now, on account of (5),

$$\frac{dx_c}{ds} = \frac{1}{\sqrt{\omega}}\frac{dx_c}{d\sigma}\,, \text{ etc.}$$

$$\frac{d^2x_c}{ds^2} = \frac{1}{\sqrt{\omega}}\frac{d}{d\sigma}\left(\frac{1}{\sqrt{\omega}}\frac{dx_c}{d\sigma}\right) = \frac{1}{\omega}\frac{d^2x_c}{d\sigma^2} - \frac{1}{2\omega^2}\frac{dx_c}{d\sigma}\frac{d\omega}{d\sigma}$$

by which, after multiplication by ω, the equation of the geodesic line becomes

$$\frac{d^2x_c}{d\sigma^2} - \tfrac{1}{2}\frac{dx_c}{d\sigma}\frac{d\,log\,\omega}{d\sigma} = -\Sigma\,(ab)\begin{Bmatrix} a\,b \\ c \end{Bmatrix}\frac{dx_a}{d\sigma}\frac{dx_b}{d\sigma}. \tag{6}$$

If, finally, we substitute the expression (3) and the value

$$\frac{d \log \omega}{d\sigma} = \Sigma (f) \frac{dx_f}{d\sigma} \frac{\partial \log \omega}{\partial x_f},$$

we are led to four equations, linear in the unknown quantities

$$\frac{\partial \log \omega}{\partial x_f}.$$

§ 4. Since we started from the assumption that EINSTEIN's theory is true, we need fear no contradiction between the different equations. But we must make sure that they are mutually independent and can give us definite values for the derivatives of $\log \omega$.

The equation which we deduce from (6) as just explained may be written in the form

$$- \Sigma (f) \frac{dx_c}{d\sigma} \frac{dx_f}{d\sigma} \frac{\partial \log \omega}{\partial x_f}$$

$$+ \Sigma (a\, b\, e)\, \gamma^{ce} \frac{dx_a}{d\sigma} \frac{dx_b}{d\sigma} \left(\gamma_{ae} \frac{\partial \log \omega}{\partial x_b} + \gamma_{be} \frac{\partial \log \omega}{\partial x_a} - \gamma_{ab} \frac{\partial \log \omega}{\partial x_e} \right) = \ldots \quad (7)$$

where the terms on the right-hand side are completely known. We have simply represented them by ....., their values being irrelevant to our purpose, which is merely to show the definiteness of the solution.

Multiplying (7) by γ_{ch} and adding the resulting equations [1]) with $c = 1,2,3,4$, we find four new equations ($h = 1, \ldots 4$)

$$\Sigma (a\, b) \left(\gamma_{bh} \frac{\partial \log \omega}{\partial x_a} - \gamma_{ab} \frac{\partial \log \omega}{\partial x_h} \right) \frac{dx_a}{d\sigma} \frac{dx_b}{d\sigma} = \ldots$$

or, again, if we add the equation that is obtained by interchanging the suffixes a and b,

$$\Sigma (a\, b)\, \Phi_{ab} \frac{dx_a}{d\sigma} \frac{dx_b}{d\sigma} = \ldots \quad (8)$$

$$\Phi_{ab} = \gamma_{ah} \frac{\partial \log \omega}{\partial x_b} + \gamma_{bh} \frac{\partial \log \omega}{\partial x_a} - 2\, \gamma_{ab} \frac{\partial \log \omega}{\partial x_h}.$$

Now, consider a definite point P and a definite suffix h. Since $\Phi_{ba} = \Phi_{ab}$ there are ten mutually independent values $\Phi_{11}, \ldots \Phi_{12}, \ldots$ and these are the same whatever be the direction of the geodesic line. The values of $\frac{dx_a}{d\sigma}$ and $\frac{dx_b}{d\sigma}$ being known, each line gives us an equation of the form (8) that must be satisfied by Φ_{ab}. Hence, if we apply (8) to ten geodesic lines passing through P, we obtain a sufficient number of equations for the determination of Φ_{ab}. There will be but one solution, if the determinant on the coefficients is not zero, a condition that will be fulfilled when the lines selected do not happen to lie on a cone of the second degree.

[1]) The relations $\Sigma (c)\, \gamma_{ch}\, \gamma^{ce} = \delta_h^e$ ($\delta_h^e = 1$ for $h = e$, and $= 0$ for $h =/= e$) may be used here.

The outcome of the calculations so far sketched is this, that for all combinations of the suffixes a, b, h, we know the expression

$$\gamma_{ah}\frac{\partial \log \omega}{\partial x_b}+\gamma_{bh}\frac{\partial \log \omega}{\partial x_a}-2\,\gamma_{ab}\frac{\partial \log \omega}{\partial x_h}=\ldots \qquad (9)$$

Let us now multiply this by γ^{ab} and add the equations which we find by giving both to a and to b the values 1, 2, 3, 4. The first term gives $\frac{\partial \log \omega}{\partial x_h}$, the second leads to the same result and from the third we find

$$-8\frac{\partial \log \omega}{\partial x_h},$$

so that the derivative $\frac{\partial \log \omega}{\partial x_h}$ becomes known. Thus, since h may be 1, 2, 3 or 4, one finds at any point of the diagram the derivatives of $\log \omega$ with respect to the coordinates. By this, apart from a constant term, $\log \omega$ becomes known. Finally, one finds the function ω and, by virtue of (1), the potentials g_{ab}, only a constant factor being left undeterminate [1]).

Here again there would be opportunities for verifications of the theory. Indeed, we have used no more than ten of the geodesics passing through P, the number of the equations (9) is far greater than four, the number of the first derivatives of $\log \omega$, and finally, when these have been determined as functions of the coordinates, the relations of the form

$$\frac{\partial^2 \log \omega}{\partial x_b\,\partial x_a}=\frac{\partial^2 \log \omega}{\partial x_a\,\partial x_b}$$

may be put to the test.

§ 5. So far we used an arbitrarily chosen system of coordinates x_a. If, instead of these, we want to introduce new coordinates x'_a, certain functions of x_a, we can follow the same method for determining the corresponding potentials g'_{ab}. We may, however, just as well take for these the values that are derived from the potentials g_{ab} first determined by means of the transformation-formulae for covariant tensors. These formulae, when applied to g_{ab}, are equivalent to the statement that ds^2

[1]) According to WEYL (Raum, Zeit, Materie, 1st ed., p. 182) the world-lines of light-signals would suffice already for this determination of the potentials. I think this cannot be said. Suppose f.i., that, after having properly chosen the coordinates, one has been able to account for the course of these lines by assuming for g_{44} and g_{ab} ($a \neq 4, b \neq 4$) certain values that are functions of the space-coordinates x_1, x_2, x_3 only, and by putting $g_{a4}=0$ for $a \neq 4$ (so that the field is a stationary one). Then, one may multiply all potentials by one and the same arbitrarily chosen function of x_1, x_2, x_3, without altering the velocities of propagation, which are determined by $ds^2=0$, and therefore without modifying the course of rays of light which follows from the velocities by means of HUYGENS' construction.

is invariant, and it is clear that, if $ds' = ds$, the new conditions $ds' = 0$, and $\delta \int ds' = 0$ for the world-lines are equivalent to the original ones $ds = 0$ and $\delta \int ds = 0$.

As to the constant factor in g_{ab} or g'_{ab}, we shall suppose it to have been chosen for the first system and to have such a value for the second that g_{ab} and g'_{ab} are related to each other in the way just mentioned.

When the potentials g_{ab} have been determined, the geometry of the extension R_4 may be completely developed on the assumption that ds represents the line-element. It will be easy f. i. to define the angle between two directions and to find the differential equation for geodesic lines. We need not speak of all this, but perhaps the following remarks will not be out of place.

1. Two line-elements (PQ and PQ') ds and $d's$ with the components dx_a and $d'x_a$ are said to be at right angles to each other when

$$\Sigma\,(ab)\, g_{ab}\, dx_a\, d'x_b = 0.$$

2. It may be inferred from this that, if the line-element QQ' is denoted by $d''s$,

$$d''s^2 = ds^2 + d's^2.$$

3. A line-element is a contravariant vector, whose direction-constants ξ^a are given by the differential coefficients $\frac{dx_a}{ds}$.

4. If a vector is displaced parallel to itself (the word "parallel" being used in the sense that was given it by LEVI CIVITA [1]), its starting point

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[1]) In order to state what is the meaning of a parallel displacement of a vector we may remark in the first place that, when, at any point P — we have two directions at right angles to each other and determined by the constants ξ^a and ξ'^a, the four quantities

$$\xi''^a = \xi^a \cos\varphi + \xi'^a \sin\varphi$$

will also satisfy the condition that is fulfilled by direction-constants, (viz. the condition

$$\Sigma\,(a\,b)\, g_{ab}\, \xi^a\, \xi^b = 1).$$

The direction which they determine is said to lie in the plane of the two given directions and to make an angle φ with the former of these.

Let P be a point in R_4 and L a geodesic line starting from it. We shall now define a parallel displacement of a vector PA, the starting point P of which moves along L.

1. If, in the first place, at the point P the vector PA is directed along L, it shall constantly be directed along that line.

2. Similarly, if originally the vector is perpendicular to the line, it shall remain at right angles to it. This, however, does not completely determine the direction of the vector when a point Q has been reached, and we therefore complete the definition as follows:

Draw from the point P a second geodesic line L' that makes an infinitely small angle with the line L and whose direction at the point P lies in the plane containing the initial direction of L and that of the vector PA. Take equal infinitely small segments PQ and PQ' on L and L'. Then the line-element QQ' will give us the direction of the vector PA after its displacement to the point Q.

moving along a line-element dx_a, the changes of its direction-constants are given by

$$d\xi^a = -\Sigma (b\,c) \left\{ \begin{matrix} b\,c \\ a \end{matrix} \right\} \xi^c\, dx_b \,. \qquad (10)$$

§ 6. Let us now imagine that not only the values of the coordinates but also those of the potentials g_{ab} are inscribed in the diagram R_4.

A physicist who wants to study phenomena as affected by the gravitational field will then be enabled, using the numbers which he sees in the diagram, to assign definite values, independent of the choice of coordinates, to lengths of lines and to intervals of time; he can express them in what may be called "invariant" measure. A first instance of this kind is the distance between two neighbouring points in the diagram, the invariant measure for which will simply be the value of ds. As a second example we may take the length of an infinitely short rod. Let L and L' be the world-lines of its extremities, A a point of the first line corresponding to the instant x_4 for which we want to evaluate the length, and B a point of L' determined by the condition that AB is perpendicular to L'. Then we shall measure the length l of the rod by [1])

$$l^2 = -AB^2, \qquad (11)$$

calculating AB^2 by means of the formula for ds^2.

The necessary calculation can be performed in the following manner. Let A' be the point of L' corresponding to the same time x_4 as A, and let B correspond to $x_4 + \tau$. Then the infinitely short time τ is determined by the condition that AB is at right angles to L' (we may just as well say, at right angles to L) and having found τ one knows $A'B^2$ and $AB^2 = AA'^2 - A'B^2$. The result is

$$l^2 = -\Sigma (a\,b)\, g_{ab} (x'_a - x_a)(x'_b - x_b) + \frac{\{\Sigma (a\,b)\, g_{ab}\, \dot{x}_a (x'_b - x_b)\}^2}{\Sigma (a\,b)\, g_{ab}\, \dot{x}_a\, \dot{x}_b} \qquad (12)$$

Repeating this construction, one can displace the vector parallel to itself over any finite part of the geodesic L,

3. If finally the vector PA in its initial position has a direction neither along the line L nor perpendicular to it, we decompose it into two components having these directions. Displacing each of them parallel to itself along the line, say to a point R, and keeping their magnitudes constant, we shall find two vectors at the point R. Compounding these we obtain a definite resulting vector and this will give us the direction of PA after a parallel displacement to the point R.

This definition of a parallel displacement along a geodesic implies the definition of such a displacement along a given infinitely short line, for such a line may always be considered as the first element of a geodesic. Proceeding by infinitely small steps, we may now also displace a vector parallel to itself along any length of an arbitrarily chosen line that is no geodesic.

Working out what has been said here, one is led to eq. (10).

[1]) If the potentials have the values that are often ascribed to them (f.i. $g_{11} = g_{22} = g_{33} = -1$, $g_{44} = c^2$, $g_{ab} = 0$ for $a \neq b$, or values little different from these) AB^2 becomes negative. In order to find a real value for l (11) has been written with the negative sign.

Here the coordinates of A are denoted by x_a and those of A' by x'_a (so that $x'_4 = x_4$). The symbols $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$ represent the components of the velocity of the first extremity of the rod, $(\dot{x}_4 = 1)$.

Owing to the way in which it has been found, the above expression (12) is invariant. It may be remarked that, if instead of the length of AB we had taken that of AA', which depends on the "simultaneous" positions of the two ends, the result would have depended on the choice of coordinates.

That l, as defined by (12), may appropriately be termed the "length" of the rod, will be clear if one remarks that, if all circumstances remain the same, l is proportional to the differences of corresponding coordinates $x'_a - x_a$, and that for a rod at rest, placed in a field characterized by

$$g_{11} = g_{22} = g_{33} = -1, \quad g_{44} = c^2, \quad g_{ab} = 0 \quad \text{for} \quad a \neq b,$$

the formula becomes

$$l^2 = (x_1' - x_1)^2 + (x_2' - x_2)^2 + (x_3' - x_3)^2.$$

§ 7. In what precedes nothing has been said of the phenomena presented by rods placed in a gravitational field; we have only adopted a rule for measuring their length. If a physicist, adhering to the theory of relativity, were able to observe all the minute effects required by this theory and wanted to account for them, he certainly would follow this rule, because it would enable him to discuss all his observations, f.i. those about the influence of temperature and of external forces, in terms that are independent of the choice of coordinates. An "ideal rod of unvariable length" would mean to him a rod whose "invariant" length l would be the same under all circumstances.

Take f.i. the case of a field, which, when x_1, x_2, x_3 are rectangular cartesian coordinates, is characterized by the potentials specified at the end of § 6, that is a field in which there are no forces of gravitation. Let a rod be placed in this field in the direction x_1 and let it move with the velocity v in that direction. Then $\dot{x}_1 = v$, $\dot{x}_2 = \dot{x}_3 = 0$, $\dot{x}_4 = 1$, so that (12) becomes

$$l^2 = (x_1' - x_1)^2 + \frac{v^2 (x_1' - x_1)^2}{c^2 - v^2} = \frac{c^2}{c^2 - v^2} (x_1' - x_1)^2.$$

Now, it would be very natural to measure the length of the rod by the difference of the simultaneous values of the coordinates x_1 and x'_1. If l is the same under all circumstances, this new length l_e (say the "euclidian" length) will change with the velocity v, according to the formula

$$l_e = \sqrt{1 - \frac{v^2}{c^2}}\, l,$$

in which one recognizes the well known contraction that is brought about by a motion of translation.

§ 8. The influence of a gravitation-field on the motion of a clock can be treated in a similar way. Let the clock be so small that we may speak of its world-line L; on this line the successive ticks will mark a series of points P, Q, R, ..., which we shall suppose to be infinitely near each other. The statement that a clock is "perfect" will have a meaning independent of the choice of coordinates, if we understand by it that the distances PQ, QR, ..., when expressed in invariant measure, will be equal for the particular clock considered, whatever be the circumstances. If the length of this distance is τ and if dx_4 is the interval of time between successive ticks, we have

$$\tau^2 = \Sigma\,(a\,b)\,g_{ab}\,dx_b\,dx_b\,,$$

or

$$\tau^2 = (g_{11}\,\dot{x}_1^2 + \ldots + 2\,g_{12}\,\dot{x}_1\,\dot{x}_2 + \ldots + 2\,g_{14}\,\dot{x}_1 + \ldots g_{44})\,dx_4^2,$$

a relation from which the value of dx_4 in different cases can be deduced.

§ 9. A rod may be conceived to have different lengths l according to the circumstances under which it is placed. A short discussion, under certain simplifying assumptions, of changes of this kind (in the case of an infinitely short rod) will be of interest with a view to a theory that has been proposed by WEYL [1]) and according to which there is a close and fundamental connexion between gravitational and electromagnetic phenomena.

The length of the rod might change with the time x_4, with the position of one of the extremities, determined by its coordinates x_1, x_2, x_3, and with the direction in which the rod is placed in the space R_3. We shall however discard this latter possibility, so that we are only concerned with variations of $x_1, \ldots x_4$. If these are infinitely small, the change produced in l may be assumed to be a homogeneous linear function of them, and if, further, we suppose it to be proportional to l itself, we may write

$$d\log l = \Sigma\,(a)\,P_a\,dx_a \qquad (13)$$

with coefficients P_a solely depending on the coordinates. P_a will be a covariant vector, because, according to the fundamental idea of EINSTEIN's theory, for a given displacement in R_4, $d\log l$ must be independent of the choice of coordinates.

Eq. (13) may be applied to any part of a world-line, say between the points C and D, which, of course, means that during a certain interval of time the position of the rod in R_3 undergoes some definite change. We shall suppose the dimensions of the line CD to be very small and we shall calculate the change of l accurately up to quantities of the second order with respect to these dimensions.

Then, if for any point E of the path we put

$$x_a = x_{a,C} + \mathfrak{x}_a\,,$$

[1]) Berl. Sitz. Ber., 1918, p. 465.

we must, in (13), replace P_a by

$$P_a + \Sigma\,(b)\,\frac{\partial P_a}{\partial x_b}\,\mathfrak{x}_b\,,$$

where we have to take, both for P_a and for its differential coefficients, their values at the point C.

The total change of $\log l$ now becomes

$$\triangle \log l = \Sigma\,(a)\,P_a \int d\mathfrak{x}_a + \Sigma\,(a\,b)\,\frac{\partial P_a}{\partial x_b}\int \mathfrak{x}_b\,d\mathfrak{x}_a\,.$$

The last term in this expression depends on the path along which the transition from C to D has taken place, and if $\triangle' \log l$ is the change corresponding to a second path $CE'D$, we have

$$\triangle \log l - \triangle' \log l = \Sigma\,(a\,b)\,\frac{\partial P_a}{\partial x_b}\int \mathfrak{x}_b\,d\mathfrak{x}_a\,. \tag{14}$$

the integral being taken along the closed path $CEDE'C$ in the direction indicated by the order of the symbols.

The integral vanishes for $a = b$ and we have $\int \mathfrak{x}_a\,d\mathfrak{x}_b = -\int \mathfrak{x}_b\,d\mathfrak{x}_a$, the sum of the two integrals being $\int d\,(\mathfrak{x}_a\,\mathfrak{x}_b) = 0$.

Thus (14) may be replaced by

$$\triangle \log l - \triangle' \log l = \tfrac{1}{2}\,\Sigma\,(a\,b)\left(\frac{\partial P_a}{\partial x_b} - \frac{\partial P_b}{\partial x_a}\right)\int \mathfrak{x}_b\,d\mathfrak{x}_a\,. \tag{15}$$

This again shows that in general the final length of the rod will be different, according to the path along which the transition from C to D has been made. If there is to be no such difference, the components of the vector P_a must depend on a potential φ, so that

$$P_a = -\frac{\partial \varphi}{\partial x_a}\,.$$

§ 10. There is a certain formal similarity between the expressions to which we have now been led and the relations which exist in an electromagnetic field.

Indeed, it is well known that the state of things in such a field can be described by means of a fourfold vector $\overline{P}_a$, the components of electric and magnetic force being given by the expressions

$$\frac{\partial \overline{P}_a}{\partial x_b} - \frac{\partial \overline{P}_b}{\partial x_a}\,, \tag{16}$$

which, taken together, form an antisymmetric covariant tensor of the second rank.

This analogy would have a deeper meaning if the two vectors P_a and $\overline{P}_a$ could be assimilated to each other, so that with a constant numerical coefficient λ,

$$P_a = \lambda\,\overline{P}_a\,. \tag{17}$$

This would mean, and it would certainly be very important, that the changes of length considered in the preceding § are the indications of an electromagnetic field and that, conversely, any electromagnetic field gives rise to changes of that kind. In particular, the electric and the magnetic force would be made responsable for the fact that the length of a rod depends on the path in R_4 that has been followed. So long, however, as these effects of an electromagnetic field have not been observed or, at all events, have not been made probable by other arguments (for one can always account for their apparent absence by a too low value of the coefficient λ), I think we had better not admit the connexion in question, confining ourselves to the introduction in electromagnetic theory of the fourfold potential and not ascribing to it any other physical meaning.

Two remarks more may be made. In the first place, if the fourfold vectors P_a and $\overline{P_a}$ really were indissolubly connected, this would amount to an action of an electromagnetic field widely different from anything that could reasonably be expected. This may be seen by taking the case of a constant electric field. In this we are concerned with one only of the components $\overline{P_a}$, namely with $\overline{P_4}$, the ordinary electrostatic potential, and the expression (13) would reduce to $P_4 dx_4$, showing that the length of a rod would, in course of time, continually and indefinitely increase or diminish. These changes might be detected by the following experiment. Of two equal rods, first juxtaposed in a region 1, one is left there, while the other is removed to a region 2 where the potential has a different value. After some time it is brought back to its original position and again compared with the first rod. The effect of these manipulations would be a difference in the two lengths that might be increased at will, simply by keeping the second rod for a longer time in the region 2.

In the second place, from electromagnetic phenomena one can deduce differences or changes only of potentials, the absolute values remaining undetermined. On the contrary, the numbers inscribed in the diagram R_4 enable us to determine in invariant measure the lengths of rods. Attending to their changes and applying eq. (13) one could obtain a knowledge of the potentials themselves.

§ 11. I shall conclude with some remarks on a generalisation of WEYL's theory that has been proposed by EDDINGTON [1]). His considerations are the more interesting because they can be developed to a certain extent without it being necessary to introduce the potentials g_{ab}.

EDDINGTON's aim is to arrive at the anti-symmetric covariant tensor F_{ab} of electric and magnetic force, at the fourfold potential on which these forces depend, and at the gravitation-potentials, making all these quantities flow from one common source. For this purpose, he begins by assigning to each point of the diagram R_4, in which coordinates, but

[1]) Proc. Roy. Soc. A. 99, p. 104 (1921); The mathematical theory of relativity, p. 213.

no values g_{ab} have been inscribed, 40 numbers. These are regarded as continuous functions of the four coordinates and are then subjected to certain mathematical operations.

The *fundamental numbers* as we may call them are represented by the symbol Γ^c_{ab}, with the relation

$$\Gamma^c_{ba} = \Gamma^c_{ab}, \tag{18}$$

by which the number of mutually independent quantities that otherwise would be 64, is reduced to 40.

Since no potentials g_{ab} have been introduced, we can speak neither of the length of a line-element, nor of the magnitude of a vector A (as we may call any line-element); there can be question only of the components dx_a or A^a. There is nothing, however, that prevents us from imagining that, when the starting point of a vector moves along a line-element dx_b, the components of the vector change in some specified manner. Making a definite assumption concerning these changes, EDDINGTON defines what he calls a parallel displacement; I shall rather say the "selected" displacement in order to keep in mind that, so long as there is no ds, there can be no question of direction-constants and of angles, nor of a parallel displacement in the sense in which the term was understood in § 5.

The fundamental numbers serve precisely for the definition of the changes in the components of a vector A^a, which accompany its displacement along a line-element dx_b, EDDINGTON's formula being

$$dA^a = -(b\,c)\,\Gamma^a_{bc}\,A^c\,dx_b\,. \tag{19}$$

Line-elements and vectors in R_4 may be conceived to remain the *same* whatever be the coordinates which one uses for the evaluation of their components, and in the definition contained in (19) it is to be understood that the element dx_b along which the displacement is effected and the vector A^a, both before and after its displacement, are always the same in this sense. Hence, equations of the form (19), but with other fundamental numbers Γ'^a_{bc} will hold after a change of coordinates. It is not difficult to find the relations between the original and the new fundamental numbers, but these transformation-formulae are found to have a form different from the one that is characteristic of tensors. In other terms, Γ^a_{bc} is not a tensor [1]).

[1]) The transformation-formula for Γ^a_{bc} is

$$\left(\frac{\partial x'_a}{\partial x_b} = \pi_{ba} \quad , \quad \frac{\partial x_a}{\partial x'_b} = p_{ab}\right)$$

$$\Gamma'^a_{bc} = -\Sigma\,(kl)\,p_{kc}\,p_{lb}\,\frac{\partial \pi_{ka}}{\partial x_l} + \Sigma\,(klm)\,p_{lb}\,p_{mc}\,\pi_{ka}\,\Gamma^k_{lm}\,. \tag{20}$$

If here, on the right-hand side we had the last term only, Γ^a_{bc} would be a tensor.

Using the relation $\frac{\partial \pi_{ka}}{\partial x_l} = \frac{\partial \pi_{la}}{\partial x_k}$, one can deduce from (20) that Γ'^a_{bc} has, like Γ^a_{bc}, the symmetry expressed in (18).

§ 12. We shall now use eq. (19) for calculating the changes $\triangle A^a$ which occur in the components of a vector, when in a succession of infinitely small selected displacements, its starting point is made to move in a closed line drawn in R_4. The dimensions of this line are supposed to be infinitely small and we shall limit ourselves to quantities that are of the second order with respect to them.

Let the motion begin at the point P, and let for any other point Q of the cycle

$$x_a - x_{aP} = \mathbf{x}_a .$$

Then we have to calculate

$$\triangle A^a = - \Sigma (b\,c) \int \Gamma^a_{bc} A^c \, d\mathbf{x}_b . \tag{21}$$

We must here keep in mind that the first factor under the sign of integration is the value of Γ^a_{bc} at the point Q and that for the second factor we must take the component A^c such as it has become when that point has been reached. It is preferable, however, to understand by these symbols the values corresponding to the fixed point P. Doing so, we must replace the first factor by

$$\Gamma^a_{bc} + \Sigma (s)\, \Gamma^a_{bc,s}\, \mathbf{x}_s ,$$

where $\Gamma^a_{bc,s}$ is the value at P of the differential coefficient of Γ^a_{bc} with respect to x_s. As to the change of A^c during the motion from P to Q, it will be sufficient to calculate it up to terms of the first order and we can therefore directly deduce it from (19), replacing the differentials dx_b by $x_{bQ} - x_{bP} = \mathbf{x}_b$ and understanding by A_c the initial values at the point P. Thus the second factor in the integral has to be replaced by

$$A^c - \Sigma (h\,i)\, \Gamma^c_{hi}\, A^i\, \mathbf{x}_h .$$

After substitution (21) becomes $\left(\int d\mathbf{x}_b = 0 \text{, and we may omit the product } \mathbf{x}_s \mathbf{x}^h \right)$

$$\triangle A^a = - \Sigma (b\,c\,s)\, \Gamma^a_{bc,s}\, A^c \int \mathbf{x}_s \, d\mathbf{x}_b + \Sigma (b\,c\,h\,i)\, \Gamma^a_{bc}\, \Gamma^c_{hi}\, A^i \int \mathbf{x}_h \, d\mathbf{x}_b .$$

We may now, in the first term, write i and h instead of c and s and then obtain a new form of the same expression by interchanging in both terms the suffixes h and b. Finally, taking half the sum of the two forms and remembering that $\int \mathbf{x}_b \, d\mathbf{x}_h = - \int \mathbf{x}_h \, d\,\mathbf{x}_b$, we find

$$\triangle A^a = \tfrac{1}{2} \Sigma (b\,i\,h)\, B^a_{ihb}\, A^i \int \mathbf{x}_h \, d\mathbf{x}_b , \tag{22}$$

where

$$B^a_{ihb} = \Gamma^a_{hi,b} - \Gamma^a_{bi,h} + \Sigma (c) \left[\Gamma^a_{bc}\, \Gamma^c_{hi} - \Gamma^a_{hc}\, \Gamma^c_{bi} \right]. \tag{23}$$

Our conclusion is therefore that in general the components of a vector will have changed when it has been carried round along a closed path

and that the changes are determined by the expressions (23). In what follows we shall be concerned with the quantities B^a_{ihb} only or rather with a tensor G_{ih} of the second rank that may be derived from them, and we shall scarcely have to think any more of the foregoing considerations, which were intended to deduce B^a_{ihb} from the fundamental numbers and to point out its geometrical meaning.

It is easily shown that B^a_{ihb} is a tensor, covariant as to the suffixes i, h, b and contravariant as to a [1]).

The covariant tensor G_{ih} that was mentioned just now is deduced from B^a_{ihb} by the operations indicated in the formula [2])

$$G_{ih} = \Sigma\,(a)\, B^a_{ih\,a}. \tag{24}$$

If the fundamental numbers are arbitrarily chosen, the tensor G_{ih} will be neither symmetric, nor antisymmetric. It may however be decomposed into a symmetric and an antisymmetric part, namely

$$R_{ih} = \tfrac{1}{2}\,(G_{ih} + G_{hi}) \quad , \quad F_{ih} = \tfrac{1}{2}\,(G_{ih} - G_{hi}).$$

It is easily seen that these are both covariant tensors and that

$$R_{hi} = R_{ih} \quad , \quad F_{hi} = -\,F_{ih}.$$

Performing the operations leading from the tensor (23) to F_{ih} and taking into account the relations (18), one finds

$$F_{ih} = \tfrac{1}{2}\,\Sigma\,(a)\,(\Gamma^a_{ah,\,i} - \Gamma^a_{ai,\,h}),$$

which shows that the tensor F depends on a fourfold potential A_k. If we put

$$A_k = \tfrac{1}{2}\,\Sigma\,(a)\,\Gamma^a_{ak}, \tag{25}$$

---

[1]) In eq. (22) we may apply to A^i the transformation-formula for line-elements. We may proceed in the same way with $\mathbf{x}_h$, because this quantity is treated as infinitely small, and with ΔA^a because it is the difference of two vectors beginning at the same point. In all these cases the quantities p_{ab} and π_{ab} which occur in the transformation-formulae may be taken such as they are at the point P. We may do the same in transforming $d\,\mathbf{x}_b$. It is true that this element lies at a certain distance from P, but the influence which this has on p_{ab} would lead to terms of an order higher than needs be considered.

Thus:

$$\Delta A'^k = \Sigma\,(a)\,\pi_{ak}\,\Delta A^a = \tfrac{1}{2}\,\Sigma\,(a\,b\,i\,h)\,\pi_{ak}\,B^a_{ihb}\,A^i \int \mathbf{x}_h\,d\mathbf{x}_b$$

$$= \tfrac{1}{2}\,\Sigma\,(a\,b\,i\,h\,l\,m\,n)\,\pi_{ak}\,p_{il}\,p_{hm}\,p_{bn}\,B^a_{ihb}\,A'^l \int \mathbf{x}'_m\,d\mathbf{x}'_n\,,$$

an expression that has the same form as (22) if we put

$$B'^k_{lm\,n} = \Sigma\,(a\,b\,i\,h)\,\pi_{ak}\,p_{il}\,p_{hm}\,p_{bn}\,B^a_{ihb}.$$

[2]) Proof that G_{ih} is a covariant tensor. It suffices to express in the components B the quantities B' occurring in

$$G'_{ih} = \Sigma\,(a)\,B'^a_{ih\,a}$$

and to use the relations

$$\Sigma\,(a)\,\pi_{ba}\,p_{ca} = \delta^c_b.$$

we may write

$$F_{ih} = \frac{\partial A_h}{\partial x_i} - \frac{\partial A_i}{\partial x_h}. \tag{26}$$

Following EDDINGTON we may now identify the quantities F_{ih} with the components of electric and magnetic force, or the components A_k with those of the fourfold electromagnetic potential.

It must be remarked, however, that the quantities A_a as defined by (25), do not constitute a tensor.

If we require them to do so, we must replace (25) by [1])

$$A_k = \tfrac{1}{2} \Sigma (a) \Gamma^a_{ak} + \frac{\partial \psi}{\partial x_k},$$

ψ being a function of the coordinates for which the transformation-formula is

$$\psi' = \psi - \tfrac{1}{2} \log p \tag{27}$$

(p is the functional determinant of the original coordinates with respect to the new ones). [1])

By the addition of the terms depending on ψ (26) is not changed.

§ 13. If h, i, j are all different, we have according to (26)

$$\frac{\partial F_{hi}}{\partial x_j} + \frac{\partial F_{ij}}{\partial x_h} + \frac{\partial F_{jh}}{\partial x_i} = 0.$$

---

[1]) Using (20) we may write

$$A'_r = \tfrac{1}{2} \Sigma (a) \Gamma'^a_{ar} + \frac{\partial \psi'}{\partial x'_r} = -\tfrac{1}{2} \Sigma (akl)\, p_{kr}\, p_{la} \frac{\partial \pi_{ka}}{\partial x_l} +$$

$$+ \tfrac{1}{2} \Sigma (aklm)\, p_{la}\, p_{mr}\, \pi_{ka}\, \Gamma^k_{lm} + \frac{\partial \psi}{\partial x'_r} - \tfrac{1}{2} \frac{\partial \log p.}{\partial x'_r}$$

But $\Sigma (a)\, p_{la}\, \pi_{ka} = \delta^l_k$ and consequently

$$\Sigma (a) \frac{\partial}{\partial x_l} (p_{la}\, \pi_{ka}) = 0.$$

By this the first term of the expression for A'_r becomes

$$\tfrac{1}{2} \Sigma (l) \frac{\partial p_{lr}}{\partial x_l} = \tfrac{1}{2} \Sigma (lu)\, \pi_{lu} \frac{\partial p_{lr}}{\partial x'_u} = \tfrac{1}{2} \Sigma (lu)\, \pi_{lu} \frac{\partial p_{lu}}{\partial x'_r} =$$

$$= \tfrac{1}{2} \Sigma (lu) \frac{1}{p} \frac{\partial p}{\partial p_{lu}} \frac{\partial p_{lu}}{\partial x'_r} = \tfrac{1}{2} \frac{\partial \log p}{\partial x'_r},$$

and if, in the third term, we write

$$\frac{\partial}{\partial x'_r} = \Sigma (m)\, p_{mr} \frac{\partial}{\partial x_m}$$

the formula becomes

$$A'_r = \Sigma (m)\, p_{mr}\, A_m,$$

showing that A_k is a covariant tensor.

It may also be noted that the transformations defined by (27) form a group.

There are four formulae of this kind and these form one group of MAXWELL's equations.

As to the other group, in which the density of electric charge and the components of the convection-current occur, these equations must necessarily contain a contravariant tensor connected with F_{ih}. A tensor F^{ab} of this kind can only be defined if we have introduced beforehand the components g_{ab}, the relation between the two tensors being expressed by the formula

$$F^{ab} = \Sigma\,(i\,h)\,g^{ai}\,g^{bh}\,F_{ih}.$$

Moreover the equations in question contain the factor $\sqrt{-g}$.

EDDINGTON has remarked, however, that the gravitation-potentials g_{ab} and all quantities that depend on them, may also be considered as derived from the fundamental numbers that have given us the components of electric and magnetic force.

Indeed, we have so far used only the antisymmetric part F_{ih} of the tensor G_{ih} and we may now have recourse to its symmetric part R_{ih}. Since g_{ih} must also be symmetric we may put

$$g_{ih} = \frac{1}{\lambda} R_{ih},$$

λ being a constant.

The effect of this will be that all quantities involved in the phenomena of gravitation and electromagnetism, namely the potentials g_{ab}, the electric and magnetic forces F_{ab} and the corresponding contravariant tensor F^{ab} have been derived from the fundamental numbers. [1])

§ 14. All that has been said in §§ 11—13 amounts to the establishment of certain rules for the mathematical operations by means of which the components of electric and magnetic force and, if so desired, the gravitation-potentials can be derived from the fundamental numbers.

Now it must be remarked that the variety of these numbers is considerably greater than that of the quantities which we want to deduce from them. Indeed, there are four components of the electromagnetic potential and ten values g_{ab}, whereas there are no less than fourty fundamental numbers. It may well be asked whether after all it would not be preferable simply to introduce the functions that are necessary for characterizing the electromagnetic and gravitational fields, without encumbering the theory with so great a number of superfluous quantities. The introduction of these could be justified, and would, of course, become very important, only if we had good grounds for thinking that some-

[1]) The only quantity occurring in the above formulae of which this cannot, as yet, be said, is the function ψ which appears in our definition of A_k. This function is to a certain extent undeterminate, the only condition being that it must transform according to eq. (27). If we put $\psi = -1/2\ log\sqrt{-g}$, an assumption that agrees with (27), ψ also will have its origin in the fundamental numbers.

thing that might sooner or later be observed lies behind their wide diversity.

It may also be remarked that, in any particular case, the fundamental numbers must be such that they lead to the really existing values of electric and magnetic force and of the gravitation-potentials. However, I have found it by no means easy to account f. i. for the values

$$g_{11} = g_{22} = g_{33} = -1, \quad g_{44} = c^2$$

by suitable assumptions concerning EDDINGTON's Γ^c_{ab}.

§ 15. Some words remain to be said about the analogy between eq. (10) and (19). The direction-constants in the former equation may be considered as the components of a vector of unit magnitude, so that, like (19), (10) expresses a rule for a certain selected displacement of a vector. In so far it is a special case of (19), Γ^a_{bc} being replaced by $\left\{ \begin{matrix} b\,c \\ a \end{matrix} \right\}$ (by which condition (18) is satisfied).

By this the tensors B^a_{ihb} and G_{ih} defined by (23) and (24) become the well known tensors connected with the curvature of R_4, and the latter of them becomes symmetric, a simplification that is important for the theory of gravitation.

In order to prove it, one has to show that the antisymmetric part (26) vanishes. This becomes clear if one takes into account that, according to (25),

$$2A_k = \Sigma(a) \left\{ \begin{matrix} a\,k \\ a \end{matrix} \right\} = \Sigma(a\,b)\, g^{ab} \left[\begin{matrix} a\,k \\ b \end{matrix} \right]$$

$$= \tfrac{1}{2} \Sigma(a\,b)\, g^{ab} \left(\left[\begin{matrix} a\,k \\ b \end{matrix} \right] + \left[\begin{matrix} b\,k \\ a \end{matrix} \right] \right) = \tfrac{1}{2} \Sigma(a\,b)\, g^{ab}\, g_{ab,k}$$

$$= \frac{1}{2g} \Sigma(a\,b) \frac{\partial g}{\partial g_{ab}} \frac{\partial g_{ab}}{\partial x_k} = \frac{1}{2g} \frac{\partial g}{\partial x_k} = \frac{\partial \log \sqrt{-g}}{\partial x_k}.$$

Mathematics. — *"Ueber Differentialinvarianten von kovarianten Tensoren"*. By R. WEITZENBÖCK.

(Communicated at the meeting of January 30, 1926).

Die Aufsuchung der Differentialinvarianten gegebener Tensoren führt auf ein kompliziertes Problem der Eliminationstheorie. Wir wollen hier zeigen, dass stets Differentialinvarianten bestehen, wenn die gegebenen Tensoren so beschaffen sind, dass wenigstens $n+1$ ihrer Komponenten willkürlich wählbar sind.

Hieran anschliessend wollen wir eine allgemeine Methode angeben für die Möglichkeit zu entscheiden ob aus einem Tensor p-ter Stufe $a_{i_1 i_2 \ldots i_p}$ ein Tensor $(p+1)$-ter Stuffe $a_{i_1 i_2 \ldots i_{p+1}}$ herzuleiten ist oder nicht, wobei die Koordinaten des zweitgenannten Tensors linear und homogen von den ersten Ableitungen der $a_{i_1 i_2 \ldots i_p}$ abhängen.

§ 1.

Es sei $a_{i_1 \ldots i_p}$ ein kovarianter Tensor p-ter Stufe bei n-Veränderlichen, also

$$f = a_{i_1 \ldots i_p}\, dx^{i_1} \ldots dx^{i_p}$$

die zugehörige Differentialform.

Eine Transformation $x_i = x_i(\bar{x}_1, x_2, \ldots, \bar{x}_n)$ induziert bei den Komponenten $a_{i_1 \ldots i_p}$ und deren partiellen Ableitungen Transformationen, die durch die sogen. "Transformationsgleichungen" dargestellt werden:

$$\left.\begin{aligned}
\bar{a}_{i_1 i_2 \ldots i_p} &= a_{k_1 k_2 \ldots k_p}\, \frac{\partial x_{k_1}}{\partial \bar{x}_{i_1}} \frac{\partial x_{k_2}}{\partial \bar{x}_{i_2}} \ldots \frac{\partial x_{k_p}}{\partial \bar{x}_{i_p}} \\
\frac{\partial \bar{a}_{i_1 \ldots i_p}}{\partial \bar{x}_{\alpha_1}} &= \ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots \\
\frac{\partial \bar{a}_{i_1 \ldots i_p}}{\partial \bar{x}_{\alpha_1}\, \partial \bar{x}_{\alpha_2} \ldots \partial \bar{x}_{\alpha_m}} &= \ldots\ldots\ldots\ldots
\end{aligned}\right\} \ldots \quad (1)$$

In der letzten dieser Gleichungen sind rechts die $(m+1)$-ten Ableitungen $\frac{\partial^{m+1} x}{\partial \bar{x}^{m+1}}$ linear enthalten.

Absolute Differentialinvarianten J von höchstens m-ter Ordnung erhält man aus (1) durch Elimination der Ableitungen $\frac{\partial x}{\partial \bar{x}}, \frac{\partial^2 x}{\partial \bar{x}^2}, \ldots, \frac{\partial^{m+1} x}{\partial \bar{x}^{m+1}}$.

Wir können nun allgemein zeigen, dass, wenn nur m genügend gross

gewählt wird, immer Differentialinvarianten J_m existieren, wenn die Anzahl N der von einander unabhängigen Komponenten $a_{i_1 i_2 \dots i_p}$ wenigstens gleich $n+1$ ist.

Zählen wir nämlich in (1) die Anzahl der willkürlich wählbaren Anfangswerte der Funktionen

$$a_{i_1 \dots i_p}, \frac{\partial a_{i_1 \dots i_p}}{\partial x_{\alpha_1}}, \dots, \frac{\partial^m a_{i_1 \dots i_p}}{\partial x_{\alpha_1} \partial x_{\alpha_2} \dots \partial x_{\alpha_m}}$$

ab, die man in einem Punkte $(x_i^{(o)})$ vorschreiben kann, dann erhalten wir die Zahl

$$N\left\{1+\binom{n}{1}+\binom{n+1}{2}+\dots+\binom{n+m-1}{m}\right\} \quad \dots \quad (2)$$

Ebenso gross ist die Zahl der untereinander bezgl. dieser Anfangswerte unabhängigen Transformationsgleichungen.

Die Anzahl der Ableitungen $\frac{\partial x}{\partial x}, \dots$ ist dagegen gegeben durch

$$n\left\{\binom{n}{1}+\binom{n+1}{2}+\dots+\binom{n+m}{m+1}\right\} \quad \dots \quad (3)$$

Nun gilt folgender Satz:

Ist $\varepsilon > 0$, dann kann man m so gross wählen, dass

$$(n+\varepsilon)\left\{1+\binom{n}{1}+\dots+\binom{n+m-1}{m}\right\} > n\left\{\binom{n}{1}+\binom{n+1}{2}+\dots+\binom{n+m}{m+1}\right\} \quad (4)$$

wird.

Beweis [1]: Durch Induktion findet man die Formel

$$1+\binom{n}{1}+\binom{n+1}{2}+\dots+\binom{n+m-1}{m}=\binom{n+m}{m}.$$

Statt (4) genügt dann der Beweis von:

$$(n+\varepsilon)\binom{n+m}{m} > n\left\{\binom{n+m+1}{m+1}-1\right\}$$

$$n+\varepsilon > n\frac{\binom{n+m+1}{m+1}}{\binom{n+m}{m}}-\frac{m}{\binom{n+m}{m}}=n\frac{n+m+1}{m+1}-\frac{n}{\binom{n+m}{m+1}}.$$

Hier hat aber die rechte Seite bei $m \to \infty$ den limes n, w.z.b.w.

Wegen $N \geqq n+1$ können wir also auch m so gross wählen, dass

$$N\left\{1+\binom{n}{1}+\binom{n+1}{2}+\dots+\binom{n+m-1}{m}\right\} > n\left\{\binom{n}{1}+\right.$$
$$\left.+\binom{n+1}{2}+\dots+\binom{n+m}{m+1}\right\}. \quad (5)$$

erfüllt ist.

[1]) Dieser Beweis stammt von dem Herrn M. J. BELINFANTE.

Der im § 1 bewiesene Satz gibt im Allgemeinen nur eine obere Grenze für die kleinste Ordnung m, wofür Differentialinvarianten existieren. Es kann nämlich die besondere Struktur der Transformationsgleichungen (1) schon auf Differentialinvarianten J_r mit $r < m$ führen.

Sind z.B. bei $n = 4$ zwei kovariante Vektoren a_i und α_i gegeben, dann ist $N = 8$ und nach (5) finden wir Drei als kleinstes m wofür die Ungleichheit

$$8\left\{1+\binom{4}{1}+\binom{5}{2}+\ldots+\binom{m+3}{m}\right\} > 4\left\{\binom{4}{1}+\binom{5}{2}+\ldots+\binom{m+4}{m+1}\right\}$$

gilt. Es existieren hier also sicher absolute Differentialinvarianten von der dritten Ordnung an. Und doch haben wir hier bereits vier absolute Invarianten erster Ordnung, nämlich die vier absoluten Invarianten der Tensoren [1])

$$a_i\ ,\ \alpha_i\ ,\ f_{ik} = rot\, a \text{ und } \varphi_{ik} = rot\, \alpha.$$

Eine Erniedrigung der Zahl m wird immer dann eintreten, wenn ein Tensor $a_{i_1 \ldots i_p}$ Symmetrieeigenschaften aufweist, die es ermöglichen, schon aus den Transformationsgleichungen für die ersten Ableitungen $\frac{\partial a_{i_1 \ldots i_p}}{\partial x_\alpha}$ die Differentialquotienten $\frac{\partial^2 x}{\partial \bar{x}^2}$ zu eliminieren; insbesonders dann, wenn diese Elimination auf einen Tensor führt, dessen Komponenten linear und homogen von diesen ersten Ableitungen $\frac{\partial a_{i_1 \ldots p}}{\partial x_\alpha}$ abhängen.

Das einfachste Beispiel für diese Möglichkeit bietet ein kovarianter, alternierender Tensor zweiter Stufe $a_{ik} = -a_{ki}$. Hier haben wir:

$$\frac{\partial \bar{a}_{ik}}{\partial \bar{x}_\alpha} = \frac{\partial a_{\lambda\mu}}{\partial x_\nu}\frac{\partial x_\lambda}{\partial \bar{x}_i}\frac{\partial x_\mu}{\partial \bar{x}_k}\frac{\partial x_\nu}{\partial \bar{x}_\alpha} + a_{\lambda\mu}\frac{\partial^2 x_\lambda}{\partial \bar{x}_i\, \partial \bar{x}_\alpha}\frac{\partial x_\mu}{\partial \bar{x}_k} + a_{\lambda\mu}\frac{\partial x_\lambda}{\partial \bar{x}_i}\frac{\partial^2 x_\mu}{\partial \bar{x}_k\, \partial \bar{x}_\alpha}.$$

Vertauschen wir hier i, k und α zyklisch und addieren, dann heben sich rechter Hand wegen $a_{ik} + a_{ki} = 0$ alle Glieder mit zweiten Ableitungen $\frac{\partial^2 x}{\partial \bar{x}^2}$ weg und wir erhalten die Transformationsgleichungen des STOKES'schen Tensors q_{ikr} von a_{ik}:

$$q_{ikr} = \frac{\partial a_{kr}}{\partial x_i} + \frac{\partial a_{ri}}{\partial x_k} + \frac{\partial a_{ik}}{\partial x_r}.$$

Allgemein lässt sich diese Erscheinung folgendermassen in Formeln einkleiden, wobei wir jetzt statt der zweiten Ableitungen $\frac{\partial^2 x}{\partial \bar{x}^2}$ Drei-Indizes-Symbole Γ^λ_{ik} verwenden wollen, die von einem willkürlichen Funda-

[1]) Vgl. G. F. C. GRISS, Dissertation, Amsterdam (1925) und ausführlicher: M. EUWE, Dissertation, Amsterdam (1926).

mentaltensor g_{ik} herrühren und nur ein rechnerisches Hilfsmittel darstellen. Es kommt dies dann darauf hinaus, das wir an Stelle der Transformationsgleichungen für die ersten Ableitungen der Tensorkomponenten die Definitionsgleichung für die kovariante Ableitung verwenden:

$$a_{i_1 \ldots i_p (\alpha)} = \frac{\partial a_{i_1 \ldots i_p}}{\partial x_\alpha} - a_{\lambda i_2 i_3 \ldots i_p} \Gamma^{\lambda}_{i_1 \alpha} - \ldots - a_{i_1 i_2 \ldots i_{p-1} \lambda} \Gamma^{\lambda}_{i_p \alpha} \quad . \quad . \quad (6)$$

Hiedurch wird die Elimination der zweiten Ableitungen $\frac{\partial^2 x}{\partial \bar{x}^2}$ aus den Transformationsgleichungen für die $\frac{\partial a_{i_1 \ldots i_p}}{\partial x_\alpha}$ zurückgeführt auf die Elimination der $\frac{\partial a_{i_1 \ldots i_p}}{\partial x^\alpha}$ aus (6). Ist letzteres möglich, dann erhalten wir einen kovarianten Tensor $(p+1)$-ter Stufe:

$$b_{i_1 \ldots i_p \alpha} = \sum_Q \eta_Q \, a_{Q(i_1 \ldots i_p (\alpha))} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

wobei $Q(i_1 \ldots i_p \, \alpha)$ eine Permutation der $p+1$ Indizes $i_1 \, i_2 \ldots i_p \, \alpha$ und η_Q einen, zu dieser Permutation Q gehörenden Zahlenkoeffizienten bedeutet. Nach (6) haben wir dann

$$b_{i_1 \ldots i_p \alpha} = \sum_Q \eta_Q \, Q\left(\frac{\partial a_{i_1 \ldots i_p}}{\partial x_\alpha}\right) - S \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

wo

$$S = \sum \eta_Q \, . \, Q \{ a_{\lambda i_2 \ldots i_p} \Gamma^{\lambda}_{i_1 \alpha} + a_{i_1 \lambda \ldots i_p} \Gamma^{\lambda}_{i_2 \alpha} + \ldots \} \quad . \quad . \quad (9)$$

identisch verschwinden muss, da (8) keine Γ^{λ}_{ik} mehr enthalten darf. Dieses Herausfallen der Γ^{λ}_{ik} muss erstens eine Folge der Symmetrie sein und zweitens bedingt sein durch die Symmetriebedingungen, denen der gegebene Tensor $a_{i_1 \ldots i_p}$ genügt.

Ist also [1])

$$\varepsilon = \sum_P \varepsilon_P \, . \, P(i_{i_1} \, i_{i_2} \ldots i_p)$$

ein Operator, der aus einem allgemeinen Tensor $A_{i_1 \ldots i_p}$ einen Tensor

$$A^*_{i_1 \ldots i_p} = \varepsilon A = \sum_P \varepsilon_P \, . \, A_{P(i_1 \ldots i_p)}$$

erzeugt, der mit $a_{i_1 \ldots i_p}$ in derselben Symmetrieklasse liegt, dann bekommen wir nach (9):

$$\varepsilon \, . \, S = \sum_P \varepsilon_P \, . \, P \{ \sum_Q \eta_Q \, . \, Q [a_{\lambda i_2 \ldots i_p} \Gamma^{\lambda}_{i_1 \alpha} + \ldots] \} = 0 \, . \quad . \quad . \quad (10)$$

Hieraus gewinnt man ein System von linearen Gleichungen zur Bestimmung der Unbekannten η_Q. Die $Q(i_1 \, i_2 \ldots i_p \, \alpha)$ sind dabei Permutationen von $p+1$ Indizes, die $P(i_1 \ldots i_p)$ dagegen sind Permutationen von p Indizes, nämlich der Indizes $i_1, i_2, \ldots, i_{j-1}, \; i_{j+1}, \ldots, i_{p,\alpha}$.

[1]) Vgl. H. WEYL, Rend. di Palermo 48, (1924), S. 29.

Mathematics. — *"Über die Differentialinvarianten eines speziellen schiefsymmetrischen Tensor p_{ik} im R_4."* By R. WEITZENBÖCK.

(Communicated at the meeting of January 30, 1926).

In einer vorhergehenden Mitteilung[1]) haben wir gezeigt, dass bei jedem Tensor mit wenigstens $n+1$ unabhängigen Komponenten von einer gewissen Ordnung m an stets Differentialinvarianten existieren.

Bei $n=4$ hat nun ein spezieller schiefsymmetrischer Tensor

$$p_{ik} = -p_{ki} \text{ mit } \mathfrak{P} = 2(p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23}) \equiv 0$$

fünf unabhängige Komponenten und die allgemeine Theorie ergibt hier $m=15$, d. h. von der 15 Ordnung an existieren sicher Differentialvarianten.

Wir zeigen in dieser Mitteilung, dass bereits bei kleinerer Differentiationsordnung m Invarianten vorhanden sind, nämlich eine relative von dritter Ordnung und eine absolute Invariante von vierter Ordnung.

§ 1.

Wir verwenden für Tensoren römische, für Tensordichten deutsche Buchstaben. Da

$$\mathfrak{P} = 2(p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23}) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

eine relative Invariante oder skalare Dichte vom Gewichte eins ist, müssen wir bei Verwendung von oberen Indices setzen:

$$p_{12} = \mathfrak{p}^{34},\ p_{13} = \mathfrak{p}^{42},\ p_{14} = \mathfrak{p}^{23},\ p_{34} = \mathfrak{p}^{12},\ p_{42} = \mathfrak{p}^{13},\ p_{23} = \mathfrak{p}^{14} \quad . \quad (2)$$

Damit erhalten wir neben den Transformationsgleichungen

$$\bar{p}_{ik} = p_{\lambda\mu}\frac{\partial x_\lambda}{\partial \bar{x}_i}\frac{\partial x_\mu}{\partial \bar{x}_k} = p_{\lambda\mu}\cdot\frac{\lambda}{\bar{\imath}}\cdot\frac{\mu}{\bar{k}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

die folgenden für die $\mathfrak{p}^{ik}$:

$$\bar{\mathfrak{p}}^{ik} = \mathfrak{p}^{\lambda\mu}\frac{\bar{\imath}}{\lambda}\frac{\bar{k}}{\mu}\Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

und für $\mathfrak{P}$ haben wir dann:

$$\mathfrak{P} = \sum_{ik} p_{ik}\,\mathfrak{p}^{ik} = \tfrac{1}{2}\,p_{ik}\,\mathfrak{p}^{ik} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Aus (3) finden wir den STOKES'schen Tensor

$$\sum_{cycl.}\frac{\partial \bar{p}_{ik}}{\partial \bar{x}_l} = \left(\sum_{cycl.}\frac{\partial p_{\lambda\mu}}{\partial x_\nu}\right)\frac{\lambda}{\bar{\imath}}\frac{\mu}{\bar{k}}\frac{\nu}{\bar{l}}; \quad . \quad . \quad . \quad . \quad . \quad (6)$$

er ist alternierend von dritter Stufe und sein eigener STOKES'scher Tensor verschwindet identisch.

[1]) Diese Proceedings, Januar (1926).

Bei $n=4$ ist ein alternierender Tensor dritter Stufe nichts anderes, als eine kontravariante Vektordichte vom Gewicht eins.

Setzen wir also

$$\mathfrak{p}^i = \frac{\partial \mathfrak{p}^{ik}}{\partial x_k}, \quad \ldots\ldots\ldots\ldots \quad (7)$$

dann ist erstens

$$\frac{\partial \mathfrak{p}^i}{\partial x_i} = 0 \quad \ldots\ldots\ldots\ldots \quad (8)$$

und zweitens haben wir an Stelle von (6) die Gleichungen

$$\bar{\mathfrak{p}}^i = \mathfrak{p}^\lambda \frac{\bar{\tau}}{\lambda} \cdot \Delta \quad \ldots\ldots\ldots\ldots \quad (9)$$

Mit Hilfe der $\mathfrak{p}^i$ bekommen wir jetzt eine kovariante Vektordichte vom Gewichte eins:

$$\mathfrak{p}_i = p_{i\lambda} \mathfrak{p}^\lambda \quad \ldots\ldots\ldots\ldots \quad (10)$$

Wegen $\mathfrak{P} \equiv 0$ haben wir dann die weiteren Beziehungen:

$$\mathfrak{p}^{ik} \mathfrak{p}_k = 0, \ \mathfrak{p}_i \mathfrak{p}^i = 0. \quad \ldots\ldots\ldots\ldots \quad (11)$$

§ 2.

Um Tensordichten zweiter Ordnung zu erhalten, berechnen wir von (10) die Rotation:

$$\pi_{ik} = \frac{\partial \mathfrak{p}_i}{\partial x_k} - \frac{\partial \mathfrak{p}_k}{\partial x_i} \quad \ldots\ldots\ldots\ldots \quad (12)$$

Dies ist weder ein Tensor, noch eine Tensordichte, denn wir haben hier

$$\bar{\pi}_{ik} = \pi_{\lambda\mu} \frac{\lambda}{\bar{\tau}} \frac{\mu}{\bar{k}} \Delta + \mathfrak{p}_\lambda \left(\frac{\lambda}{\bar{\tau}} \frac{\Delta}{\bar{k}} - \frac{\lambda}{\bar{k}} \frac{\Delta}{\bar{\tau}} \right) \quad \ldots\ldots\ldots \quad (13)$$

also z.B. Invarianz bei $\Delta = \text{const}$. Wir nennen π_{ik} einen "Pseudotensor" und wollen Pseudotensoren durch griechische Buchstaben andeuten.

Multiplizieren wir (13) mit

$$\bar{\mathfrak{p}}_r = \mathfrak{p}_\nu \frac{\nu}{\bar{r}} \Delta \quad \ldots\ldots\ldots\ldots \quad (14)$$

und addieren zyklisch bezgl. i,k und l, dann fallen rechter Hand die Ableitungen von Δ heraus und wir bekommen die Transformationsgleichungen einer alternierenden Tensordichte dritter Stufe. Sie kann ersetzt werden durch eine kontravariante Vektordichte $\mathfrak{q}^i$ vom Gewichte drei:

$$\bar{\mathfrak{q}}^i = \mathfrak{q}^\lambda \frac{\bar{\tau}}{\lambda} \Delta^3 \quad \ldots\ldots\ldots\ldots \quad (15)$$

Hiebei ist

$$\mathfrak{q}^i = \pi^{i\lambda} \mathfrak{p}_\lambda \quad \ldots\ldots\ldots\ldots \quad (15^a)$$

und $\pi^{12} = \pi_{34}$, $\pi^{13} = \pi_{42}$ u.s.w. gesetzt.

Aus den π_{ik} finden wir einen Pseudoskalar

$$\Pi = 2(\pi_{12}\pi_{34} + \pi_{13}\pi_{42} + \pi_{14}\pi_{23}) = \sum_{ik} \pi_{ik}\pi^{ik} = \tfrac{1}{2}\pi_{ik}\pi^{ik} \quad . \quad . \quad (16)$$

dessen Transformierte durch

$$\overline{\Pi} = \Pi \cdot \Delta^3 - 2\Delta^2 q^\lambda \frac{\partial \Delta}{\partial x_\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

gegeben wird. Wir haben auch:

$$\Pi = -\frac{\partial q^i}{\partial x_i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

denn:

$$\frac{\partial q^i}{\partial x_i} = \frac{\partial (\pi^{i\lambda} p_\lambda)}{\partial x_i} = \frac{\partial \pi^{i\lambda}}{\partial x_\lambda} p_\lambda + \pi^{i\lambda} \frac{\partial p_\lambda}{\partial x_i},$$

also wegen $\dfrac{\partial \pi^{i\lambda}}{\partial x_\lambda} = 0$:

$$= \pi^{i\lambda} \frac{\partial p_\lambda}{\partial x_i} = \tfrac{1}{2}\pi_{i\lambda}\left(\frac{\partial p_\lambda}{\partial x_i} - \frac{\partial p_i}{\partial p_\lambda}\right) = \tfrac{1}{2}\pi^{i\lambda}\pi_{\lambda i} = -\Pi.$$

Weiters gilt

$$p_{ik}\pi^{ik} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Beweis:

$$p_{ik}\pi^{ik} = p^{ik}\pi_{ik} = p^{ik}\left(\frac{\partial p_i}{\partial x_k} - \frac{\partial p_k}{\partial x_i}\right) = 2p^{ik}\frac{\partial p_i}{\partial x_k} = 2\left[\frac{\partial (p^{ik}p_i)}{\partial x_k} - p_i \frac{\partial p^{ik}}{\partial x_k}\right],$$

also nach (11) und (7):

$$= -2p_i p^i = 0.$$

Wir setzen:

$$\omega_i = \pi_{ik} p^k; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

ω_i ist ein kovarianter Pseudovektor, denn wir haben:

$$\overline{\omega_i} = \omega_\lambda \frac{\partial x_\lambda}{\partial \overline{x_i}} \Delta^2 + \Delta p_\lambda \frac{\partial x_\lambda}{\partial \overline{x_i}} p^\mu \frac{\partial \Delta}{\partial x_\mu} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Nach (15a) wird weiter:

$$p_{ik} q^k = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Formen wir $p^{ik}\omega_k = p^{ik}\pi_{k\lambda}p^\lambda$ um nach der Formel:

$$a^{ik} b_{kr} \xi_i \eta^r = -b^{ik} a_{kr} \xi_i \eta^r + \tfrac{1}{2} a^{ik} b_{ik} \xi_r \eta^r \quad . \quad . \quad . \quad . \quad (23)$$

dann wird:

$$p^{ik}\omega_k = -\pi^{ik} p_{k\lambda} p^\lambda + \tfrac{1}{2} p^{ik}\pi_{ik}\omega_\lambda,$$

also nach (19):

$$p^{ik}\omega_k = -\pi^{ik}p_k = -q^i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Auf analoge Weise bekommen wir

$$\pi_{ik} q^k = \tfrac{1}{2}\Pi \cdot p_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$\pi^{ik}\omega_k = \tfrac{1}{2}\Pi \cdot p^i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Schliesslich ergeben sich noch die Beziehungen:

$$\mathfrak{p}_i\, \mathfrak{q}^i = 0,\ \mathfrak{p}^i\, \omega_i = 0,\ \mathfrak{q}^i\, \omega_i = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

§ 3.

Aus den Vektordichten $\mathfrak{p}^i$ und $\mathfrak{q}^i$ finden wir nun die alternierende Tensordichte vom Gewichte vier:

$$\mathfrak{r}^{ik} = (\mathfrak{p}\,\mathfrak{q})^{ik} = \mathfrak{p}^i\, \mathfrak{q}^k - \mathfrak{p}^k\, \mathfrak{q}^i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Hieraus ergeben sich die Beziehungen

$$\mathfrak{r}^{ik}\, \mathfrak{p}_k = 0,\ \mathfrak{r}^{ik}\, \omega_k = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

und wenn wir wieder $\mathfrak{r}_{12} = \mathfrak{r}^{34}$, $\mathfrak{r}_{13} = \mathfrak{r}^{42}$, etc. setzen:

$$\mathfrak{r}_{ik}\, \mathfrak{p}^k = 0,\ \mathfrak{r}_{ik}\, \mathfrak{q}^k = 0,\ \mathfrak{r}_{ik}\, \mathfrak{p}^{ik} = 0 \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Dass auch

$$\mathfrak{r}^{ik}\, \pi_{ik} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

ist, zeigt man folgendermassen:

$$\mathfrak{r}^{ik}\, \pi_{ik} = \pi_{ik}\, (\mathfrak{p}\,\mathfrak{q})^{ik} = 2\, \pi_{ik}\, \mathfrak{p}^i\, \mathfrak{q}^k = -2\, \omega_k\, \mathfrak{q}^k\,,$$

also $= 0$ nach (27).

Aus den $\mathfrak{r}^{ik}$ leiten wir jetzt den kontravarianten Pseudovektor dritter Ordnung her:

$$\zeta^i = \frac{\partial \mathfrak{r}^{ik}}{\partial x_k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Seine Transformationsgleichungen sind:

$$\overline{\zeta}^i = \zeta^\lambda\, \frac{\partial \overline{x}_i}{\partial x_\lambda}\, \Delta^4 + 3\, \Delta^2\, \mathfrak{r}^{\lambda\mu}\, \frac{\partial \overline{x}_i}{\partial x_\lambda}\, \frac{\partial \Delta}{\partial x_\mu} \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Seine Divergenz verschwindet:

$$\frac{\partial \zeta^i}{\partial x_i} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Wir haben dann nach (27) und $\mathfrak{p}_i\, \mathfrak{g}^i = 0$, $\mathfrak{g}_i\, \mathfrak{q}^i = 0$:

$$\left.\begin{aligned} \mathfrak{p}_i\, \zeta^i = \mathfrak{p}_i\, \frac{\partial\, (\mathfrak{p}\,\mathfrak{q})^{ik}}{\partial x_k} = \frac{\partial\, (\mathfrak{p}_i\, (\mathfrak{p}\,\mathfrak{q})^{ik})}{\partial x_k} - (\mathfrak{p}\,\mathfrak{q})^{ik}\, \frac{\partial \mathfrak{p}_i}{\partial x_k} = \\ = -\tfrac{1}{2}\, (\mathfrak{p}\,\mathfrak{q})^{ik}\, \pi_{ik} = -\pi_{ik}\, \mathfrak{p}^i\, \mathfrak{q}^k = \omega_k\, \mathfrak{q}^k = 0 \end{aligned}\right\} \quad . \quad . \quad (35)$$

und wieder nach (27):

$$\mathfrak{r}^{ik}\, \omega_k = (\mathfrak{p}\,\mathfrak{q})^{ik}\, \omega_k = \mathfrak{p}^i\, \mathfrak{q}^k\, \omega_k - \mathfrak{p}^k\, \omega_k\, \mathfrak{q}^i = 0 \quad . \quad . \quad . \quad (36)$$

Multiplizieren wir also (21) mit (33), dann erhalten wir:

$$\overline{\zeta}^i\, \overline{\omega}_i = \zeta^\lambda\, \omega_\lambda\, .\, \Delta^6,$$

d.h.

$$\mathfrak{T} = \zeta^i\, \omega_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

ist eine skalare Dichte vom Gewichte sechs und der Ordnung drei.

Dass $\mathfrak{T}$ nicht immer identisch Null ist, ersieht man aus dem folgenden Beispiel:

$$p_{12}=x_1^4,\quad p_{13}=0,\quad p_{14}=x_2,\quad p_{34}=x_2^2,\quad p_{42}=x_4,\quad p_{23}=-x_1^4 x_2.$$

Hier erhalten wir:

$$\begin{cases} p^1=2x_2 \\ p^2=0 \\ p^3=-1 \\ p^4=4x_1^3 x_2 \end{cases} \qquad \begin{cases} p_1=4x_1^3 x_2^2 \\ p_2=-x_1^4 x_2-4x_1^3 x_2 x_4 \\ p_3=4x_1^3 x_2^3 \\ p_4=-x_2^2 \end{cases}$$

$$\begin{cases} \pi_{12}=\pi^{34}=12x_1^3 x_2+12x_1^2 x_2 x_4 & \pi_{34}=\pi^{12}=0 \\ \pi_{13}=\pi^{42}=-12x_1^2 x_2^3 & \pi_{42}=\pi^{13}=-2x_2+4x_1^3 x_2 \\ \pi_{14}=\pi^{23}=0 & \pi_{23}=\pi^{14}=-12x_1^3 x_2^2 \end{cases}$$

$$\begin{cases} q^1=4x_1^3 x_2^4+16x_1^6 x_2^4 \\ q^2=-12x_1^2 x_2^5 \\ q^3=-4x_1^3 x_2^3-16x_1^6 x_2^3-12x_1^2 x_2^3 x_4 \\ q^4=12x_1^6 x_2^4 \end{cases}$$

$$\begin{cases} r^{12}=r_{34}=-24x_1^2 x_2^6 & r^{34}=r_{12}=4x_1^6 x_2^4+64x_1^9 x_2^4+48x_1^5 x_2^4 x_4 \\ r^{13}=r_{42}=-4x_1^3 x_2^4-16x_1^6 x_2^4-24x_1^2 x_2^4 x_4 & r^{42}=r_{13}=-48x_1^5 x_2^6 \\ r^{14}=r_{23}=8x_1^6 x_2^5-64x_1^9 x_2^5 & r^{23}=r_{14}=-12x_1^2 x_2^5 \end{cases}$$

$$\begin{cases} \zeta^1=-144x_1^2 x_2^5 \\ \zeta^2=48x_1 x_2^6 \\ \zeta^3=72x_1^2 x_2^4+144x_1^5 x_2^4+ \\ \qquad\qquad +48x_1 x_2^4 x_4 \\ \zeta^4=-7.48.x_1^5 x_2^5+9.64.x_1^8 x_2^5 \end{cases} \qquad \begin{cases} \omega_1=12x_1^2 x_2^3 \\ \omega_2=-4x_1^3 x_2^2-24x_1^2 x_2^2 x_4- \\ \qquad\qquad -16x_1^6 x_2^2 \\ \omega_3=24x_1^2 x_2^4 \\ \omega_4=0 \end{cases}$$

Also wird:

$$\mathfrak{T}=-8.24.x_1^4 x_2^8 (1-14x_1^3).$$

§ 4.

Da $\mathfrak{T}$ das Gewicht sechs hat, sind für $\mathfrak{T}\not\equiv 0$

$$p^i=\mathfrak{p}^i\,\mathfrak{T}^{-1/6} \quad \ldots\ldots\ldots \quad (38)$$

und

$$q^i=\mathfrak{q}^i\,\mathfrak{T}^{-1/2} \quad \ldots\ldots\ldots \quad (39)$$

zwei kontravariante Vektoren, womit man weitere Differential-invarianten finden kann [1]). So erhalten wir z. B. durch die sogen. Klammerausdrücke einen kovarianten Vektor von vierter Ordnung:

$$r^i = p^\lambda \frac{\partial q^i}{\partial x_\lambda} - q_\lambda \frac{\partial p^i}{\partial x_\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Schliesslich liefert die Divergenz der Vektordichte

$$\mathfrak{d}^i = \mathfrak{q}^i \, \mathfrak{T}^{-1/3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

eine absolute Differentialinvariante von vierter Ordnung:

$$J = \frac{\partial \mathfrak{d}^i}{\partial x_i} \mathfrak{T}^{-1/6} = -\mathfrak{T}^{-3/2} \left[\Pi \mathfrak{T} + \tfrac{1}{3} \mathfrak{q}^i \frac{\partial \mathfrak{T}}{\partial x_i} \right] \quad . \quad . \quad . \quad . \quad (42)$$

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[1]) Vgl. G. F. C. GRISS, Dissertation, Amsterdam (1925) S. 50 ff.

Chemistry. — *"On Crystalforms of Derivatives of Ethoxy- and Trimethyl-Benzophenone"*. By Prof. F. M. JAEGER.

(Communicated at the meeting of January 30, 1926).

In the following paper are published the results of a series of crystal-measurements with some derivatives of *Benzophenone*. They were given to me, as already many others of this kind, for this purpose by Dr. P. J. MONTAGNE of Leyden, and they are the last members of the long series of compounds synthesized by this author in the course of his investigations, which a short time ago were suddenly ended by his too early death.

A. 4-Ethoxy-Benzophenone[1]): $C_6H_4(OC_2H_5).CO.C_6H_5$; Mpt.: 42°—46°,5 C.

From benzene + ligroine this compound is obtained in big, flat, colour-

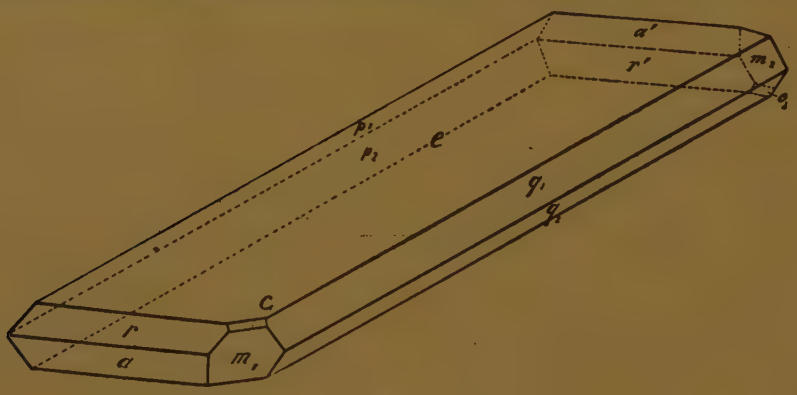

Fig. 1.

less crystals, which have meltingpoints situated between 42° C. and 46°,5 C., without being, however, different in any other respect. (Fig. 1).

Monoclinic-sphenoidal.

$a : b : c = 0{,}7427 : 1 : 1{,}6049$;

$\beta = 87° 58'$.

Forms observed: $c = \{001\}$, predominant, very lustrous, but often yielding multiple reflections and occasionally striated parallel to the *a*-axis; $q = \{011\}$, well developed, in most cases giving dull images, equally large as $p = \{0\bar{1}1\}$, which form, however, possesses lustrous faces; $r = \{101\}$,

[1]) P. J. MONTAGNE, Receuil d. Trav. d. Chim. Pays-Bas, **41**, 709, (1922).

very lustrous and well developed, broader than $a = \{100\}$; $m = \{110\}$, distinct, very lustrous and yielding sharp reflections; $o = \{111\}$, narrow, well reflecting, but often absent.

Angular values:	*Observed:*	*Calculated:*
$c : a = (001) : (100) =$	* 87° 58′	—
$c : q = (001) : (011) =$	* 58 3½	—
$a : m = (100) : (110) =$	36 35	—
$c : r = (001) : (101) =$	63 37	63° 30′
$r : a = (101) : (100) =$	24 20	24 28
$q : q = (011) : (01\bar{1}) =$	63 53	63 53
$c : m = (001) : (110) =$	88 19	88 20
$c : o = (001) : (111) =$	68 20	68 10½
$o : m = (111) : (110) =$	19 49	19 47½
$p : r = (0\bar{1}1) : (101) =$	76 13	76 20½
$r : m = (101) : (110) =$	42 52	43 2
$m : q = (110) : (01\bar{1}) =$	60 51	60 37½
$a : p = (100) : (0\bar{1}1) =$	91 6	91 4½
$a : q = (100) : (011) =$	88 54	88 5½
$c : p = (001) : (0\bar{1}1) =$	58 3½	58 3½
$m : q = (110) : (011) =$	58 45	58 39

Distinct cleavage parallel to {001}.

The plane of the optical axes is {010}; inclined dispersion.

B. 3-Bromo-4-Ethoxy-Benzophenone[1]: $C_6H_3Br.(OC_2.H_5).CO.C_6H_5$; Mpt.: 102°,2 C.

From benzene + ligroine this substance was obtained in rectangular and beautifully crystallized plates, which are colourless, perfectly transparent and well developed.

Rhombic-bipyramidal.

$a : b : c = 0{,}7935 : 1 : 0{,}2691.$

Forms observed: $a = \{100\}$, predominant, very lustrous, but giving mostly double images; $m = \{110\}$, $r = \{101\}$ and $b = \{010\}$, all about equally large, yielding sharp reflections; $s = \{102\}$, very narrow, but well measurable; $o = \{121\}$, extremely small and faint, often totally absent. (Fig. 2).

Fig. 2.

[1]) P. J. MONTAGNE, loco cit., p. 714.

Angular values:	*Observed:*	*Calculated:*
$a : m = (100) : (110) =$	* 38° 26′	—
$a : r = (100) : (101) =$	* 71 16	—
$m : b = (110) : (010) =$	51 34	51° 34′
$r : s = (101) : (\bar{1}02) =$	9 2	9 5
$s : s = (102) : (\bar{1}02) =$	19 25	19 17½
$r : m = (101) : (110) =$	75 27	75 26
$r : o = (101) : (121) =$	27 6	27 1½
$a : o = (100) : (121) =$	73 35	73 22
$b : o = (010) : (121) =$	62 51	62 58½

Perfectly cleavable parallel to {100}.

The plane of the optical axes is {001}, with the *a*-axis as first bisectrix; feeble birefringence, small optical axial angle.

The isomeric *2-Bromo-4′-Ethoxy-Benzophenone*: $C_6H_4Br \,.\, CO \,.\, C_6H_4 (OC_2H_5)$, Mpt.: 79°,5 C. has already been described before [1]). Although this compound has the same symmetry, its crystals show no distinct crystallographical relation with those here described, neither in their angular values nor in their parameters.

C. *3-5-Dibromo-Ethoxy-Benzophenone*[2]): $C_6H_2Br_2(OC_2H_5).CO.C_6H_5$; Mpt.: 83°,5 C.

From ligroine in big colourless, strongly refracting and very lustrous crystals (Fig. 3).

Monoclinic-prismatic.

$a : b : c = 1{,}0901 : 1 : 0{,}8561$;

$\beta = 59°{,}41'$

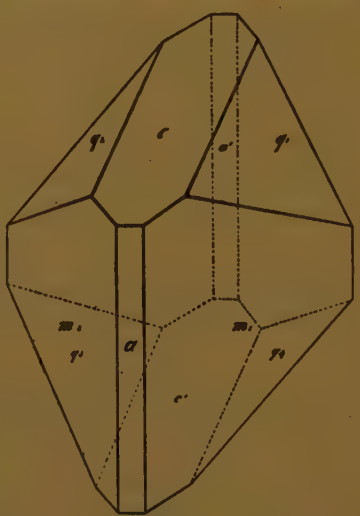

Fig. 3.

Forms observed: $m = \{110\}$, large and lustrous, yielding sharp reflections; $a = \{100\}$, narrower than *m*, but very lustrous; $q = \{011\}$, well developed and giving very sharp images; $c = \{001\}$, narrower than *m* and *q*, broader than *a*, often rough and yielding multiple reflections. The habit is short-prismatic, almost isometrical, occasionally with a slight elongation parallel to the *a*-axis. Some crystals show *q* predominant, *m* small, *c* narrow, thus having a distinctly flat, pyramidal shape.

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[1]) F. M. JAEGER, Zeits. f. Kryst., **56**, 59, (1921).

[2]) P. J. MONTAGNE, loco cit., p. 715.

Angular values:	*Observed:*	*Calculated:*
$q:q = (011):(0\bar{1}1) =$*	72° 56′	—
$m:m = (110):(1\bar{1}0) =$*	86 31	—
$m:c = (110):(001) =$*	68 26	—
$c:q = (001):(011) =$	36 28	36° 28′
$a:m = (100):(110) =$	43 15½	43 15½
$c:a = (001):(\bar{1}00) =$	59 47	59 41
$m:m = (110):(\bar{1}10) =$	93 29	93 29
$q:m = (011):(\bar{1}10) =$	83 26	83 35½
$q:m = (011):(110) =$	45 43	45 20

No distinct cleavage could be found.

The plane of the optical axes is {010}; on {100} one axis is observed at the border of the field. Inclined dispersion.

D. 4-Nitro-2′-4′-6′-Trimethylbenzophenone[1]: $C_6H_4(NO_2) . CO . C_6H_2(CH_3)_3$.

This substance crystallizes from ethyl-acetate in big, pale yellowish, strongly lustrous crystals. Occasionally individuals are met with, whose faces are less perfect, thus giving multiple images. The measurements were made with the best crystals. (Fig. 4).

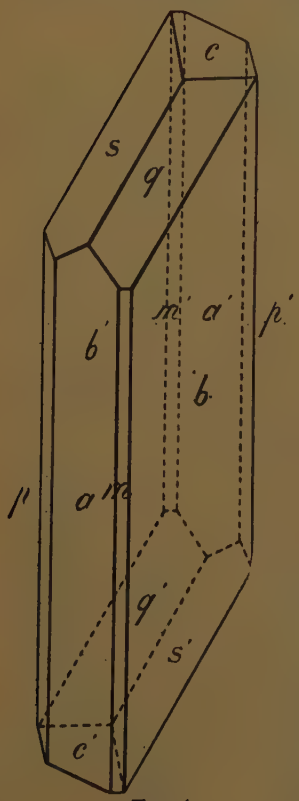

Fig. 4.

Triclinic-pinacoidal.

$a:b:c = 1{,}6303:1:0{,}6347$;
$A = 100°26′$; $\alpha = 99°41′$.
$B = 85°43$; $\beta = 91°39$.
$C = 56°33$; $\gamma = 56°45½$.

Forms observed: $b = \{010\}$, very lustrous and in most cases predominant; $a = \{100\}$, narrower, giving good reflections; $\omega = \{\bar{1}11\}$ and $o = \{111\}$, yielding excellent reflections; $c = \{001\}$, also excellently reflecting and well developed; $m = \{110\}$ and $p = \{1\bar{1}0\}$, narrow, often giving good images; $t = \{120\}$, often absent, but otherwise narrow and well reflecting. The habit of the crystals is mostly thick-prismatic and flattened parallel to b. Often, however, the crystals are peculiarly distorted: e.g. they are prismatic parallel to the edge $o:\omega$. In other individuals b and t are very narrow, but m is very large in this case.

[1]) Prepared by MONTAGNE. Although the structure mentioned here is most probable, it must yet be proved. As the author's death made this impossible, the formula given here should be considered as preliminary.

Angular values:	*Observed:*	*Calculated:*
$a:m=(100):(110)=$	* 86° 19′	—
$m:b=(110):(010)=$	* 37 8	—
$c:b=(001):(010)=$	* 79 34	—
$a:c=(100):(001)=$	* 94 17	—
$\omega:b=(1\bar{1}1):(0\bar{1}0)=$	* 52 7	—
$b:p=(0\bar{1}0):(1\bar{1}0)=$	20 36	20° 42½′
$p:a=(1\bar{1}0):(100)=$	35 56	35 50½
$b:o=(010):(111)=$	57 15	57 12
$o:\omega=(111):(1\bar{1}1)=$	70 38	70 41
$a:\omega=(100):(1\bar{1}1)=$	53 27	53 33½
$a:o=(100):(111)=$	88 31	88 21
$m:o=(110):(111)=$	51 5	51 7½
$c:o=(001):(111)=$	29 30	29 35
$c:\omega=(001):(1\bar{1}1)=$	52 12	52 10
$b:t=(010):(120)=$	16 50	16 51
$t:m=(120):(1\bar{1}0)=$	20 18	20 17
$c:m=(001):(\bar{1}\bar{1}0)=$	99 25	99 17½
$c:p=(001):(\bar{1}10)=$	80 59	80 50
$p:\omega=(1\bar{1}0):(1\bar{1}1)=$	46 49	47 0

No distinct cleavage was found. The extinction on {010} is almost normally orientated, i. e. almost parallel and perpendicular to the edge $b:m$.

E. 3-5-Dibromo-2′-4′-6′-Trimethyl-Benzophenone[1]): $C_6H_3Br_2 . CO . C_6H_2(CH_3)_3$.

From ligroine large, very lustrous and almost colourless crystals were obtained, which show a strong variation of their habit. Also the angular values appear to oscillate not inappreciably with different individuals. The following data were obtained by means of the best developed crystals. (Fig. 5).

Monoclinic-prismatic.

$a:b:c=0{,}5291:1:0{,}5873$;

$\beta=70°\ 14\frac{1}{2}'$.

Forms observed: $c=\{001\}$, broad and lustrous, ordinarily predominant, but occasionally showing also smaller dimensions; $o=\{121\}$ and $m=\{110\}$, both generally equally large

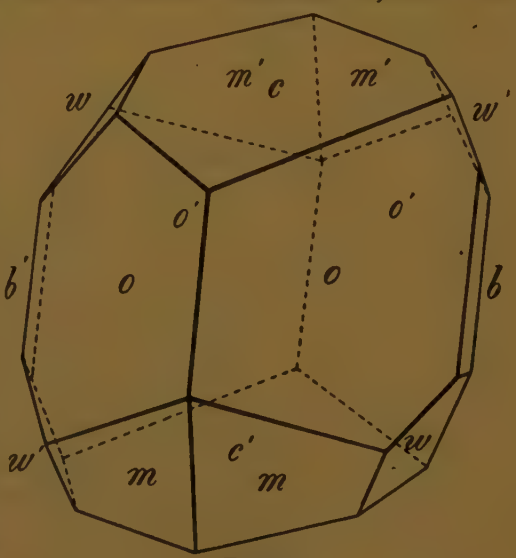

Fig. 5.

[1]) P. J. MONTAGNE: Also the paper concerning this and the following substance has not been published. However, the constitution of both these compounds has been made certain.

and very lustrous, but occasionally o is strongly predominant and m very narrow; $\omega = \{\bar{1}41\}$, smaller than o, exceptionally even very insignificant, but always yielding good reflections; $b = \{010\}$, very narrow, often absent, but e. g. in the needle-shaped crystals very well reflecting; $q = \{011\}$, rather small, giving good images, but often absent. The habit is thick and tabular parallel to $\{001\}$ or elongated in the direction of an edge $b : o$. Also other combinations occur.

Angular values:	*Observed:*	*Calculated:*
$c : o = (001) : (12\bar{1}) =$	* 47° 53′	—
$m : m = (110) : (1\bar{1}0) =$	* 52 56½	—
$c : m = (001) : (110) =$	* 72 23	—
$o : o = (121) : (\bar{1}2\bar{1}) =$	65 20	65° 14½′
$o : m = (121) : (110) =$	29 22	29 15
$o : \omega = (121) : (14\bar{1}) =$	58 40	58 38
$o : \omega = (1\bar{2}1) : (\bar{1}4\bar{1}) =$	64 25	64 22
$\omega : o = (\bar{1}4\bar{1}) : (\bar{1}2\bar{1}) =$	48 15	48 22½
$o : m = (1\bar{2}1) : (110) =$	67 12	67 16
$c : \omega = (001) : (\bar{1}4\bar{1}) =$	75 36	75 40
$c : q = (001) : (011) =$	28 59	28 56
$b : q = (010) : (011) =$	61 1	61 4
$o : q = (121) : (0\bar{1}1) =$	70 59	70 58
$q : \omega = (0\bar{1}1) : (\bar{1}4\bar{1}) =$	49 50	49 48
$q : o = (1\bar{2}1) : (0\bar{1}1) =$	32 4	32 1
$o : b = (121) : (010) =$	57 20	57 23

No distinct cleavage was found.

On $\{001\}$ there is a diagonal extinction; on m and b it is oblique with respect to the vertical axis.

F. 4-4′-4″-4‴-*Tetrabromo-Benzopinacone* [1]: $(C_6H_4Br)_2\,C\,.\,OH\,.\,.\,C.\,OH.\,(C_6H_4Br)_2 + 1$ *Ethyl-acetate.*

From ethylacetate this substance crystallizes in thick, colourless, prismatic crystals, containing 1 molecule of the solvent. Already at a slight increase of temperature they turn opaque by loss of ethylacetate. The clear, transparent crystals appear to be well built and show constant angles.

Monoclinic-prismatic.

$$a : b : c = 1{,}7540 : 1 : 0{,}4636;$$

$$\beta = 81°\ 41'.$$

[1]) Prepared by P. J. MONTAGNE; concerning this compound must be applied what has been remarked in the footnote on the preceding page.

Forms observed: $a = \{100\}$, large, often curved and yielding multiple reflections; $m = \{110\}$, somewhat broader than $b = \{010\}$, both giving very sharp images; $o = \{111\}$, broad and lustrous; $\omega = \{\bar{1}11\}$, much smaller than o, but also well reflecting. The habit is elongated parallel to the *c*-axis and flattened parallel to {100}. (Fig. 6).

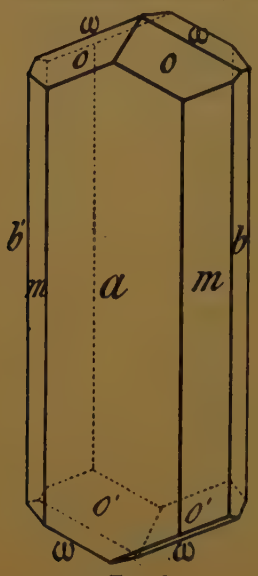

Fig. 6.

Angular values:	*Measured*:	*Calculated*:
$a : m = (100) : (110) =$	* 60° 3′	—
$a : \omega = (\bar{1}00) : (\bar{1}11) =$	* 83 44	—
$b : o = (010) : (111) =$	* 67 18	—
$b : \omega = (010) : (\bar{1}11) =$	65 23	65° 17⅓′
$o : o = (111) : (1\bar{1}1) =$	45 24	45 24
$\omega : \omega = (\bar{1}\bar{1}1) : (\bar{1}11) =$	49 14	49 25⅓
$b : m = (010) : (110) =$	29 57	29 57
$o : \omega = (111) : (\bar{1}11) =$	26 55	26 54
$a : o = (100) : (111) =$	69 22	69 26

No distinct cleavage was found.

On {100} the extinction was normal. The plane of the optical axes is {010}; on (100) and (101) one of the optical axes can be seen excentrically. The dispersion is inclined and rather strong.

Groningen, Laboratory for Inorganic and Physical Chemistry of the University.

Chemistry. — "*The Crystalforms of Some Organic Nitrogen-compounds.* By Prof. F. M. JAEGER.

(Communicated at the meeting of January 30, 1926).

§ 1. In this paper the crystalforms are described of the following fifteen nitrogen-derivatives, which have been investigated during the last few years: *p-Amino-acetophenone; Acetyl-phenyl-urethane; Acetyl-α-amino-pyridine; α-Amino-pyridine-urethane; Acetyl-chinine; Tartronaminic-acid; d-Tartramidic-acid; Trihydrazide-hydrochloride of Tricarballylic acid; Amido-sulphonic acid; Phtalimidine; Ammonium-phtalimidine-acetate; Ethylenediamine-hydrochloride; p-Nitro-mono-propyl-aniline; Benzoylcyano-acetic methyl-ether* and *m-Nitro-benzoylcyano-acetic methyl-ether.* Also the description is added here of *Triethylphosphine-carbodisulphide,* the remarkable addition-product of *carbondisulphide*: CS_2 and $P(C_2H_5)_3$.

§ 2. *p-Amino-Acetophenone* [1]: $CH_3 . CO . C_6H_4(NH_2)$; Mpt: 106° C. From benzene in large, flat, pale yellowish, transparent crystals; from a mixture of alcool + chloroform in rectangular plates, elongated either parallel to the *b*-, or to the *a*-axis (fig. 1).

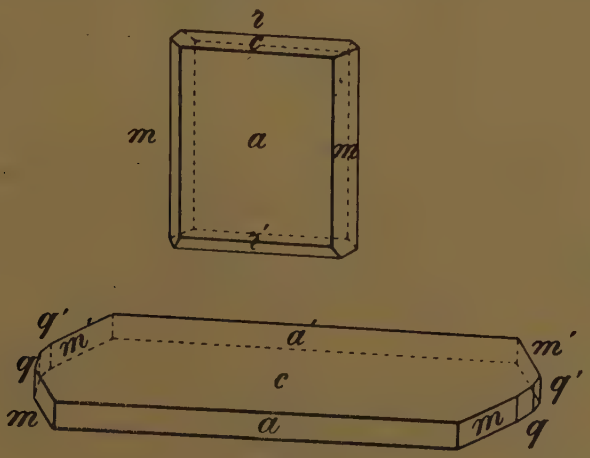

Fig. 1.

Monoclinic-prismatic.

$a:b:c = 1{,}7447:1:1{,}6662;$

$\beta = 73° 16'.$

Forms observed: $c = \{001\}$, mostly predominant and lustrous; $a = \{100\}$, well developed; $m = \{110\}$, broader than $p = \{120\}$; $r = \{\bar{1}01\}$, well developed and yielding good reflections. The habit is often elongated parallel to the *b*-axis, with $\{001\}$ and $\{100\}$ predominant, and in the last case laso prismatic along the *c*-axis.

[1] V. B. DREWSEN, Lieb. Ann. der Chemie, **212**, 162, (1882).

Angular values:	*Observed:*	*Calculated:*
$a:r = (100):(\bar{1}01) =$	* 55° 9′	—
$a:m = (100):(110) =$	* 59 6	—
$c:a = (001):(100) =$	* 73 16	—
$c:r = (001):(\bar{1}01) =$	51 33	51° 35′
$p:p = (120):(\bar{1}20) =$	33 55	33 19
$p:m = (120):(110) =$	14 3	14 14½
$c:p = (001):(120) =$	85 20½	85 16
$c:m = (001):(110) =$	81 35	81 29

In the prism-zone the angular values vary not inappreciably. No distinct cleavability. The optical axial plane is {010}; the first bisectrix is almost perpendicular to {001}. The angle of the optical axes is evidently very small; weak and inclined dispersion.

§ 3. *Acetyl-phenyl-Urethane* [1]): $C_6H_5 . N\langle^{CO . CH_3}_{CO . OC_2H_5}$; Mpt.: 59° C.

From ligroine in colourless, well-developed crystals. (Fig. 2).

Rhombic-bipyramidal.

$a:b:c = 1{,}2323:1:15141$.

Forms observed: $c = \{001\}$, large and lustrous; $m = \{110\}$, also well developed and giving sharp images; $a = \{100\}$ and $b = \{010\}$, generally almost equally large; $o = \{211\}$, well reflecting, mostly smaller, but occasionally much larger than m. The habit is short-prismatic with tabular form parallel to {001}, or that of thick plates parallel to {001}.

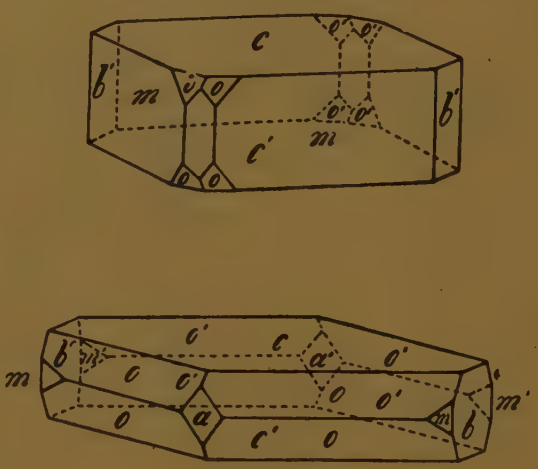

Fig. 2.

Angular values:	*Observed:*	*Calculated:*
$b:m = (010):(110) =$	* 39° 3½′	—
$c:o = (001):(211) =$	* 55 17	—
$a:o = (100):(211) =$	45 32	45° 30′
$b:o = (010):(211) =$	64 25	64 28
$o:o = (211):(2\bar{1}1) =$	51 10	51 4
$a:m = (100):(110) =$	50 56½	50 56½
$m:m = (110):(1\bar{1}0) =$	101 53	101 53
$m:o = (110):(211) =$	39 12	39 7
$o:o = (211):(21\bar{1}) =$	69 28	69 26
$m:o = (1\bar{1}0):(211) =$	83 55	83 54½

No distinct cleavage could be found.

[1]) D. R. Nijk, Rec. d. Trav. chim. d. Pays-Bas, **39**, 701, (1920).

The plane of the optical axes is probably {100}; the *c*-axis is first bisectrix. The apparent axial angle is only small; the dispersion is weak.

§ 4. *Acetyl-α-Amino-pyridine* [1]): $C_5H_4N.NH.CO.CH_3$: *Mpt*: 71° C. From benzene + ligroine in colourless, very lustrous crystals. (Fig. 3).

Monoclinic-prismatic.

$a:b:c = 1{,}4939:1:2{,}0719;$

$\beta = 89°14\frac{1}{2}'.$

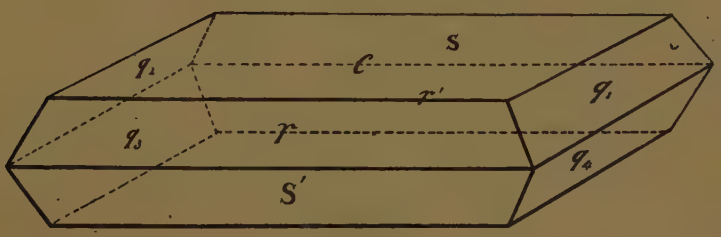

Fig. 3.

Forms observed: $c = \{001\}$, predominant, yielding good reflections; $r = \{100\}$ and $s = \{\bar{1}01\}$, equally broad, well developed and giving sharp images; $q = \{011\}$, broad and lustrous. The habit is that of thick tables parallel to {001}, with elongation in the direction of the *b*-axis.

Angular values:	*Observed:*	*Calculated:*
$c:r = (001):(101) =$	* 53° 42½′	—
$c:s = (001):(\bar{1}01) =$	* 54 42½	—
$c:q = (001):(011) =$	* 64 14	—
$r:s = (101):(10\bar{1}) =$	71 35	71° 35′
$q:q = (011):(01\bar{1}) =$	51 32	51 32
$r:q = (101):(011) =$	74 59	75 5½

Perfectly cleavable parallel to {010}.

§ 5. *α-Amino-pyridine-Urethane* [2]): $C_5H_4N.NH.CO.OC_2H_5$; *Mpt*: 105°,5 C.

From ether this substance is obtained in well developed, but very soft and plastic crystals; they are colourless and commonly possess a prismatic or parallelopipedal shape. By the translation-planes present the crystals can only be manipulated, if the necessary precautions be taken; the angular values vary not inappreciably in the different zones.

Monoclinic-prismatic.

$a:b:c = 0{,}7946:1:1{,}3552;$

$\beta = 73° 21'.$

1) E. DINGEMANSE and J. P. WIBAUT, Rec. d. Trav. Chim. d. Pays-Bas, **42**, 240, (1923),

2) E. DINGEMANSE and J. P. WIBAUT, Rec. d. Trav. Chim. d. Pays-Bas, loco cit.

Forms observed: $m = \{110\}$, large and lustrous; $a = \{100\}$, narrower than m, often absent, but giving good reflections; $c = \{100\}$, well develoed and yielding very good reflections; $r = \{101\}$, well reflecting, much broader than $s = \{108\}$, which form is often absent.

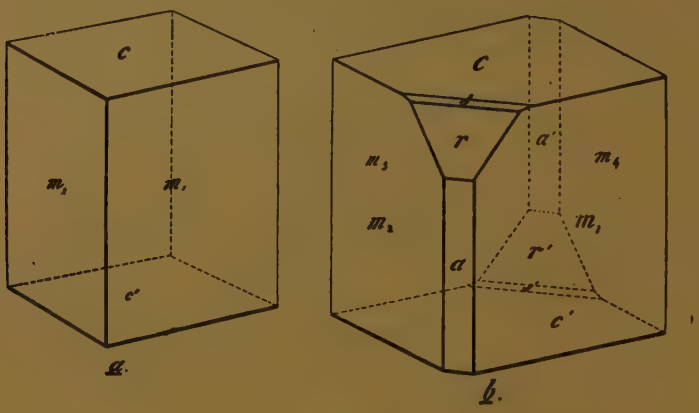

Fig. 4.

Angular values:	*Observed:*	*Calculated:*
$m : m = (110) : (1\bar{1}0) =$	* 74° 34′	—
$c : m = (001) : (110) =$	* 76 49½	—
$c : r = (001) : (101) =$	47 40	—
$c : s = (001) : (108) =$	10 40	10° 54′
$a : m = (100) : (110) =$	37 23	37 17
$a : s = (100) : (108) =$	63 0	62 27
$r : s = (101) : (108) =$	36 45	36 46

Cleavage perfect parallel to {101}.
On {001} diagonal, on {110} oblique extinction.

§ 6. *Acetyl-chinine*: [1]) $C_{20}H_{23}(CO \cdot CH_3) \cdot N_2O_2$; *Mpt*: 116° C.
This compound crystallizes at room-temperature from ether in splendid, colourless and transparent crystals, having a high lustre and measuring often several cm. in length. (Fig. 5).

Rhombic-bisphenoidal.

$$a : b : c = 1{,}1142 : 1 : 0{,}6119.$$

Forms observed: $a = \{100\}$ and $m = \{110\}$, about equally large and very lustrous; they yield excellent reflections, although on a often multiple images. Further: $b = \{010\}$, much narrower, often only with a single plane, well reflecting; $s = \{201\}$ and $r = \{101\}$, both giving excellent

[1]) L. SEEKLES, Dissertation, Leiden, (1922), p. 33.

images, s mostly much smaller than r; $o = \{\bar{1}11\}$, large and lustrous, almost always showing a single plane; $\omega = \{111\}$, small, dull, but well measurable. The crystals are well built and have constant angular values. Their habit is thick-prismatic and elongated parallel to the c-axis or flattened parallel to two planes of m and a.
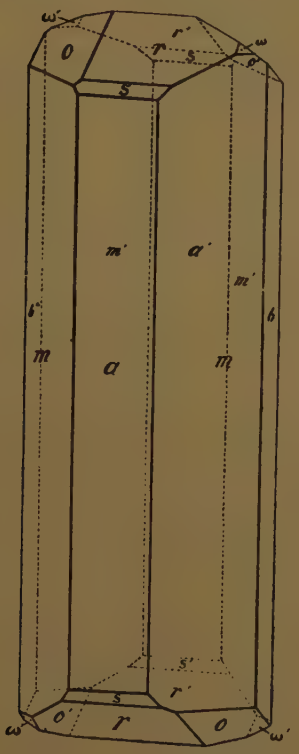
Fig 5.

Angular values:	*Observed:*	*Calculated:*
$a : m = (100) : (110) =$	* 48° 5½′	—
$a : s = (100) : (201) =$	* 42 19	—
$b : m = (010) : (110) =$	41 58½	41° 54½′
$m : m = (110) : (\bar{1}10) =$	83 57	83 49
$s : r = (201) : (101) =$	18 49	18 54½
$r : r = (101) : (\bar{1}01) =$	57 59	57 33
$a : o = (\bar{1}00) : (\bar{1}11) =$	64 48	64 53½
$o : \omega = (\bar{1}11) : (111) =$	50 20	50 13
$\omega : a = (111) : (100) =$	64 48	64 53½
$m : \omega = (110) : (111) =$	50 28	50 33¾
$m : s = (110) : (201) =$	60 19	60 24
$s : o = (\bar{2}01) : (\bar{1}11) =$	33 38	33 31¼
$o : m = (\bar{1}11) : (110) =$	86 18	86 4¾
$b : \omega = (010) : (111) =$	61 36	61 47½
$\omega : r = (111) : (101) =$	28 23	28 12½
$m : r = (110) : (101) =$	71 21	71 15
$b : o = (010) : (\bar{1}11) =$	61 32	61 47½
$o : r = (\bar{1}11) : (\bar{1}01) =$	28 28	28 12½

No distinct cleavage was found.

In solution the substance is levogyrate: $[\alpha]_D = -120,°8$.

§ 7. *Tartronaminic Acid* [1]: $NH_2 . CO . CH(OH) . COOH$; *Mpt*: 160° C. with decomposition.

From water this compound crystallizes commonly in colourless flat needles, which are arranged in spherolithes, but rarely also in big, colourless, prismatic crystals. They are well built and allow accurate measurements.

Monoclinic-prismatic.

$a : b : c = 3,0003 : 1 : 2,4063$;

$\beta = 64°19'$.

Forms observed: $c = \{001\}$ and $a = \{100\}$, both giving very sharp images and almost equally broad; $o = \{111\}$,

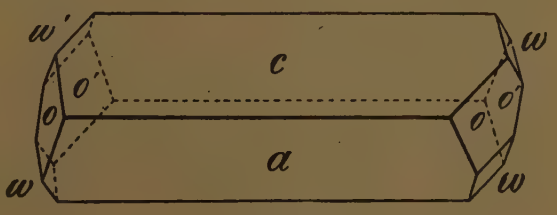

Fig. 6.

[1]) N. MENSCHUTKIN, Lieb. Ann. der Chemie, 182, 82, (1876).

splendidly reflecting and much better than $\omega = \{\bar{1}11\}$, which form shows only narrow and dull faces. The habit is thick-prismatic parallel to the *b*-axis.

Angular values:	*Observed:*	*Calculated:*
$a : c = (100) : (001) =$	* 64° 19′	—
$c : o = (001) : (111) =$	* 59 31	—
$a : o = (100) : (111) =$	* 62 16	—
$o : o = (111) : (\bar{1}1\bar{1}) =$	70 34	70° 23′
$c : \omega = (001) : (\bar{1}11) =$	74 8	74 4¼
$\omega : o = (\bar{1}11) : (\bar{1}1\bar{1}) =$	46 21	46 24¾
$a : \omega = (\bar{1}00) : (\bar{1}11) =$	80 59	81 9
$\omega : o = (\bar{1}11) : (111) =$	36 40	36 35

No distinct cleavage could be observed.

On *a* and *c* normal extinction. The plane of the optical axes is {010}; inclined dispersion. One branch of the hyperbola is on {001} visible at the border of the field of the microscope.

§ 8. *Dextrogyrate Tartramidic Acid* [1]: $COOH.CHOH.CHOH.CO.NH_2$; *Mpt*: 172° C.

Crystals of the shape reproduced in Fig. 7 were obtained from a solution in water. They are colourless, highly lustrous and yield excellent images.

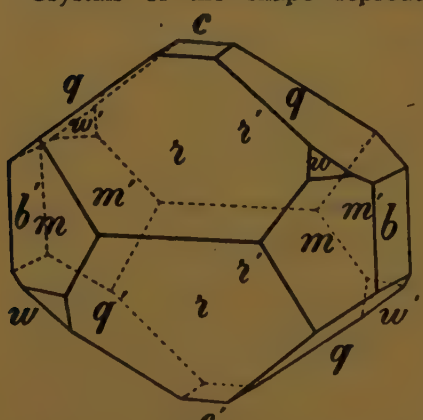

Fig. 7.

Rhombic-bisphenoidal; (pseudo-tetragonal).

$a : b : c = 0{,}7352 : 1 : 0{,}7393$.

Forms observed: $m = \{110\}$, $r = \{201\}$ and $q = \{011\}$, all about equally well developed, very lustrous and giving good reflections; $\omega = \{221\}$, somewhat smaller, but yielding sharp images, as also $b = \{010\}$; $c = \{001\}$, distinct, but a little duller. The crystals are nearly isometrical, with a slight flattening in the direction of the *a*-axis.

Angular values:	*Observed:*	*Calculated:*
$b : m = (010) : (110) =$	* 53° 40⅔′	—
$q : q = (011) : (0\bar{1}1) =$	* 72 57	—
$r : r = (201) : (20\bar{1}) =$	52 58	52° 52⅓
$c : r = (001) : (201) =$	63 31	63 33⅔
$b : \omega = (010) : (221) =$	56 43	56 38½

[1]) Comp. e. g.: L. PASTEUR, Ann. de Chim. et Phys., (3), 38, 454, (1853); P. GROTH, *Chem. Kryst.*, III, 302, 308. The preparation here investigated was made by Dr. R. A. WEERMAN Diss. Delft, (1916), p. 109.

Angular values:	*Observed:*		*Calculated:*	
$\omega : r = (221) : (201) =$	33	18	33	$21\frac{1}{2}$
$m : m = (110) : (1\bar{1}0) =$	72	$39\frac{1}{2}$	72	$38\frac{2}{3}$
$b : q = (010) : (011) =$	53	$31\frac{1}{2}$	53	$31\frac{1}{2}$
$c : q = (001) : (011) =$	36	$28\frac{1}{2}$	36	$28\frac{1}{2}$
$m : q = (110) : (011) =$	49	$37\frac{1}{2}$	49	$37\frac{1}{2}$
$r : q = (201) : (011) =$	69	8	69	1
$m : r = (110) : (201) =$	43	41	43	$49\frac{2}{3}$

No distinct cleavage was found.

On {010} and {001} normal extinction. The plane of the optical axes is {100}, with the *c*-axis probably as first bisectrix. PASTEUR observed in the crystals of the corresponding levogyrate isomeride, instead of ω, a form $o = \{\bar{2}21\}$.

§ 9. *Triethylphosphine-Carbodisulphide*[1]: $P(C_2H_5)_3 . CS_2$; Mpt: 122° C. This compound crystallizes from alcohol in flat, blood-red, very brittle needles.

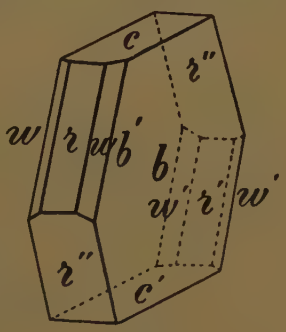

Fig. 8.

Monoclinic-prismatic (pseudo-rhombic).

$$a : b : c = 0{,}6306 : 1 : 0{,}9716;$$

$$\beta = 89° 51\frac{1}{2}'.$$

Forms observed: $b = \{010\}$, strongly predominant and, just as $r = \{101\}$ and $r'' = \{\bar{1}01\}$, yielding very sharp images; $\omega = \{111\}$, narrow, but well measurable; $c = \{001\}$, lustrous and well reflecting.

Angular values:	*Observed:*		*Calculated:*
$r : r'' = (101) : (\bar{1}01) =^*$	114°	2′	—
$b : \omega = (010) : (111) =^*$	62	9	—
$c : r = (001) : (101) =^*$	56	55	—
$r : \omega = (101) : (111) =$	27	51	27° 51′
$r : b = (101) : (010) =$	90	0	90°

Perfectly cleavable parallel to {010} and {101}. The plane of the optical axes is practically parallel to {001}; the first bisectrix is perpendicular to {010} and has a positive character. Very strongly dichroitic: on *b* for vibrations parallel to {001} dark-red, for those perpendicular to this, orange-yellow.

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[1]) According to J. P. WIBAUT (Rec. d. Trav. Chim. d. Pays-Bas, **44**, (1925), 239) the meltingpoint of 95° C., as given in literature, must be considered as wrong. With some other choice of the parameters, the crystals investigated would be identical with those investigated by SELLA (Rend. Acad. Lincei, (2), **20**, 361, (1863).

§ 10. *Trihydrazide-hydrochloride- of Tricarballylic Acid*[1]: $C_3H_5(CO.NH.NH_2)_3$, $3HCl$.

From alcool by slow evaporation in vacuo beautiful, colourless crystals were obtained.

Cubic-hexakisoctahedral.

Well built octahedra, with $(111):(1\bar{1}1) = 70°33'$. The small corrosion-figures on the faces of {111} were hexagons with their edges parallel to those of the octahedron. The crystals are isotropous, without circular polarisation.

§ 11. *Amidosulphonic Acid*: $NH_2.SO_3H$.

From a solution in water big, colourless, splendidly built and perfectly transparent crystals are obtained, as reproduced in Fig. 9.

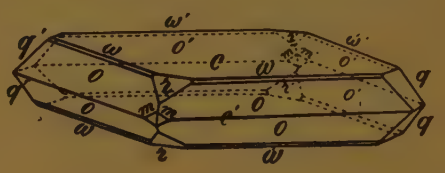


Fig. 9

Rhombic-bipyramidal;
(*pseudo-tetragonal*).
$a:b:c = 0{,}8720:1:0{,}8774$.

Forms observed: $c = \{001\}$, large and very lustrous; $o = \{111\}$, also well developed and yielding sharp reflections; $r = \{102\}$ and $q = \{021\}$, almost equally large and well reflecting; $m = \{210\}$, appreciably smaller, but distinct and exactly measurable; $\omega = \{114\}$, very narrow and giving somewhat duller images. The habit is almost isometrical, with flattening parallel to {001}; often r and ω show only half the number of faces, the crystals therefore showing an apparent monoclinic symmetry. However, the crystals are really pseudo-tetragonal: the angle $(001):(101) = 45°\,10'\frac{1}{2}$, so that the b-axis would be the vertical axis of the apparently tetragonal individual; $a:c$ becomes then about: 1 : 1,142. The extinction on {210} and the other optical properties prove, however, the true rhombic symmetry of the crystals.

Angular values:	*Observed:*	*Calculated:*
$c:o = (001):(111) = $*	53° 9¾′	—
$m:m = (210):(\bar{2}10) = $*	47 7	—
$c:q = (001):(021) =$	60 20¼	60° 19½′
$q:q = (021):(02\bar{1}) =$	59 19	59 21
$c:r = (001):(102) =$	26 57½	26 42¼
$o:o = (111):(1\bar{1}1) =$	63 30	63 28⅔
$o:o = (111):(11\bar{1}) =$	73 40½	73 40½
$c:\omega = (00\bar{1}):(11\bar{2}) -$	18 45	18 27

No distinct cleavage could be found.

The plane of the optical axes is {001}, with the b-axis as first bisectrix. The apparent axial angle is large, the dispersion weak.

[1]) Th. Curtius and A. Hesse, Journ. für prakt. Chem., **62**, 236, (1900).

Evidently the crystals are identical with those described by A. FOCK (Zeits. f. Kryst., **14**, 531, (1888)), where {010}={001} in our crystals.

§ 12. *Phtalimidine*: $C_6H_4 <^{CO}_{CH_2}> NH$; Mpt: 150°C.

From a mixture of chloroform and carbondisulphide flat parallelogram-shaped crystals were obtained, as reproduced in fig. 10.

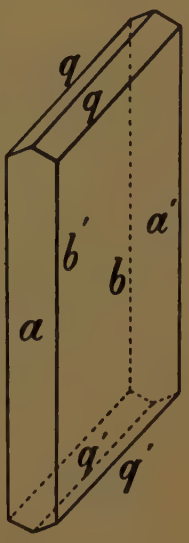


Fig. 10.

Monoclinic-prismatic.

$b : c = 1 : 0{,}4913$;

$\beta = 79° 20^2/_3'$.

Forms observed: $b = \{010\}$, strongly predominant and yielding sharp reflections, as also $a = \{100\}$; $q = \{100\}$, well developed.

Angular values:	*Observed:*	*Calculated:*
$a : q = (100) : (011) =$	* 80° 25′	—
$b : q = (010) : (011) =$	* 64 13½	—
$q : q = (011) : (0\bar{1}1) =$	51 33¼	51° 33′
$a : b = (100) : (010) =$	90 0	90°

Perfectly cleavable parallel to {010}; distinctly parallel to {100}.

The plane of the optical axis is probably parallel to {010}. On *b* one of the optical principal sections makes an angle of $19^1/_2°$ with the *c*-axis in the posterior quadrant.

§ 13. *Ammonium-Phtalimidine-acetate*: $C_6H_4 <^{CO}_{CH_2}> N\text{-}CH_2\text{-}COO(NH_4) + 2\,H_2O$.

This salt was deposited from an aqueous solution in the form of big, colourless cristals of the shape reproduced in Fig. 11.

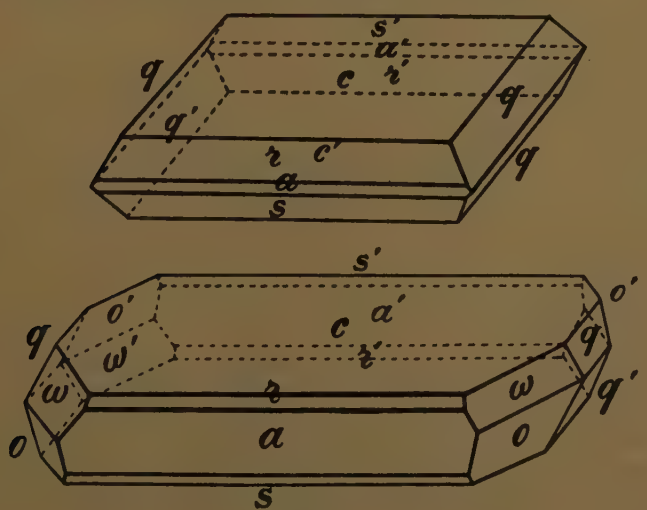


Fig. 11.

Monoclinic-prismatic.

$a : b : c = 0{,}8837 : 1 : 1{,}4301 :$

$\beta = 77°46'.$

Forms observed: $c = \{001\}$, strongly predominant and giving excellent images; $q = \{011\}$, broad, also very lustrous; $r = \{101\}$, broad reflecting; $a = \{100\}$, narrow and dull, but occasionally broader than r and yielding good reflexes; $s = \{\bar{1}01\}$, narrower than r, well measurable; $\omega = \{122\}$, small, well reflecting; $o = \{\bar{3}62\}$, larger than ω and yielding rather good images. The habit is tabular parallel to $\{001\}$ with elongation in the direction of the b-axis.

Angular values:	*Observed:*	*Calculated:*
$c : r = (001) : (101) =$	* 49° 40′	—
$r : a = (101) : (100) =$	* 28 6	—
$c : q = (001) : (011) =$	* 54 25	—
$a : s = (100) : (10\bar{1}) =$	35 2	34° 48′
$s : c = (10\bar{1}) : (00\bar{1}) =$	67 27	67 26⅓
$q : q = (011) : (01\bar{1}) =$	71 12	71 10
$r : q = (101) : (011) =$	67 54	67 52½
$s : q = (10\bar{1}) : (01\bar{1}) =$	76 55	77 6
$a : q = (100) : (011) =$	82 48	82 55
$c : \omega = (001) : (122) =$	54 4	53 53⅓
$\omega : o = (122) : (3\bar{6}2) =$	30 30	30 21
$o : c = (3\bar{6}2) : (00\bar{1}) =$	95 46	95 45⅓
$a : \omega = (100) : (122) =$	58 41	59 5
$\omega : q = (122) : (011) =$	23 53	23 50

No distinct cleavability.

§ 14. *Ethylenediamine-hydrochloride*: $C_2H_4(NH_2)_2$, $2\,HCl$.

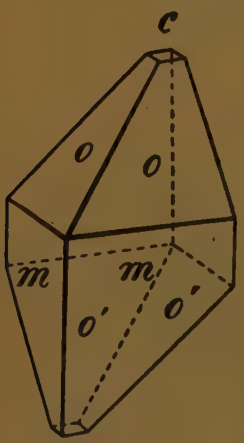


Fig. 12.

From water at room-temperature, this compound is obtained in splendid, colourless, perfectly transparent and strongly refracting crystals. They allow exact measurements to be made. (Fig. 12).

Monoclinic-prismatic.

$a : b : c = 1{,}4496 : 1 : 0{,}6597 ;$

$\beta = 88° 26^3/_4'.$

Forms observed: $m = \{110\}$, very lustrous and yielding excellent images, as also $o = \{111\}$; $c = \{001\}$, small, well measureable; $\omega = \{122\}$, in most cases only rudimentary, with strongly curved faces; often even totally absent. The habitus is that of flat, kite-shaped crystals or of steep needles or long prisms.

Angular values:	*Observed:*	*Calculated:*
$m:m=(110):(1\bar{1}0)=$*	110° 47′	—
$c:m=(001):(110)=$*	89 7	—
$o:o=(111):(1\bar{1}1)=$*	61 26	—
$m:o=(1\bar{1}0):(111)=$	77 56	77° 58½′
$m:o=(110):(111)=$	50 40	50 45½
$c:o=(001):(111)=$	38 12	38 21½
$m:\omega=(\bar{1}10):(122)=$	71 19	71 1

It seems that no distinct cleavage is present here.

§ 15. *p-Nitro-mono-Propyl-Aniline*: $C_6H_4(NO_2).NH(C_3H_7)$.

From alcohol this compound is deposited in the form of elongated, deep yellow, flat prisms with oblique terminal faces. They are highly lustrous, have a lilac surface-hue and yield multiple images. (Fig. 13).

Triclinic-pinacoidal.

$a:b:c=0{,}6135:1:0{,}2992;$
$A=59°\ 38';\ \alpha=64°\ 42'.$
$B=125\ 21\ ;\ \beta=121°\ 17\frac{1}{3}'.$
$C=72\ 38\ ;\ \gamma=89°\ 30'.$

Forms observed: $b=\{010\}$, predominant and, like $a=\{100\}$ and $c=\{001\}$, all yielding very sharp reflexes; $m=\{230\}$, $p=\{130\}$ and $t=\{\bar{2}30\}$, all about equally broad and well reflecting; $q=\{011\}$, distinctly developed, lustrous; $s=\{0\bar{1}1\}$, small and faintly reflecting. The habit is elongated in the direction of the *c*-axis, with flattening parallel {010}.

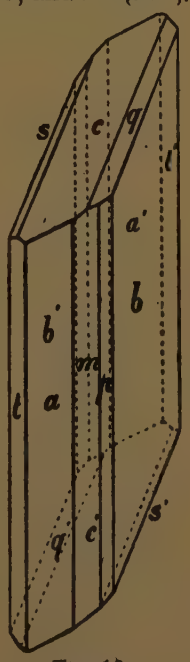


Fig. 13.

Angular values:	*Observed:*	*Calculated:*
$a:c=(100):(001)=$*	54° 39′	—
$a:b=(100):(010)=$*	107 22	—
$b:c=(010):(001)=$*	120 22	—
$c:q=(001):(011)=$*	14 13	—
$p:b=(130):(010)=$*	33 31	—
$b:t=(0\bar{1}0):(\bar{2}30)=$	39 22	39° 15′
$a:t=(100):(\bar{2}30)=$	33 16	33 23
$a:m=(100):(230)=$	48 1	48 17
$m:p=(230):(130)=$	25 40	25 34
$b:q=(010):(011)=$	106 9	106 9
$c:s=(001):(0\bar{1}1)=$	11 3	11 3½
$s:b=(0\bar{1}1):(0\bar{1}0)=$	48 35	48 34½
$c:m=(001):(230)=$	82 42	82 54

On {010} oblique extinction; the optical principal section makes, on

this face, an angle of about 12° with the direction of the *c*-axis. Strongly dichroitic: orange-yellow-pale yellow.

§ 16. *Methylether of Benzoyl-cyano-acetic Acid*: $(CN).CH(CO.C_6H_5).CO.OCH_3$; Mpt: 75°,2 C.

The compound was prepared from *benzoylchloride* and the *sodium*-salt of *methyl-cyano-acetate* at a low temperature. [1])

From absolute alcohol the substance crystallizes at 60° C in beautiful, pale yellow, very lustrous prisms, which are often flattened parallel to two opposite planes of *m*.

Rhombic-bipyramidal.

$a : b : c := 1{,}1433 : 1 : 0{,}7052.$

Forms observed: $m = \{110\}$, predominant and yielding excellent reflections; $q = \{011\}$, large and very lustrous; $o = \{111\}$, smaller than

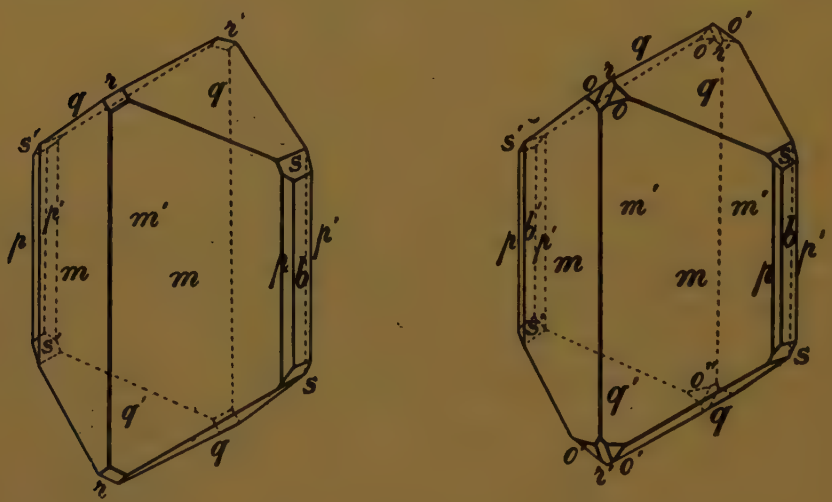


Fig. 14.

q, often apparently in two bisphenoids: the corrosion-figures on *m* and *o* prove, however, beyond doubt that the symmetry is really bipyramidal. Further: $b = \{010\}$, very narrow, often hardly visible and yielding only weak images; $p = \{120\}$, well developed and lustrous, often absent; $r = \{101\}$, small and dull; $s = \{021\}$, also subordinate, but giving good reflections. The habit is short-prismatic parallel to the *c*-axis. A number of combinations were observed, dependent on circumstances during the crystallisation: e.g..: {110}, {011} and {111}; {110}, {011}, {111}, {021} and {010}; {110} and {111}; {110} {111} and {011}; {110}, {111}, {011}, {021}, {010}; {101} and {120}; etc. Compare Fig. 14.

[1]) See also: L. BARTHE, Compt. rend. Paris, **106**, 1416, (1888); A. HALLER, ibid., **101**, 1270, (1883).

Angular values:	*Observed:*	*Calculated:*
$o : o = (111) : (\bar{1}\bar{1}1) =$	* 86° 16′	—
$m : q = (110) : (011) =$	* 64 17½	—
$m : m = (110) : (1\bar{1}0) =$	97 35½	97° 39′
$q : q = (011) : (0\bar{1}1) =$	70 18	70 23
$m : o = (1\bar{1}0) : (111) =$	46 52	46 52
$o : o = (111) : (1\bar{1}1) =$	61 54	61 57
$o : r = (111) : (101) =$	30 57	30 58½
$b : p = (010) : (120) =$	23 44	23 37½
$p : m = (120) : (110) =$	17 32	17 33
$m : r = (110) : (101) =$	69 39½	69 47
$r : q = (101) : (011) =$	45 57½	45 55½
$b : s = (010) : (021) =$	35 29	35 20
$s : q = (021) : (011) =$	19 22	19 28½

No distinct cleavability was stated.

§ 17. *Methyl-ether of m-Nitro-benzoyl-cyano-acetic Acid*: $C_6H_4(NO_2)CO.CH(CN).CO.OCH_3$.

The substance was prepared in an analogous way as the former, *m-Nitrobenzoylchloride* being used in this case.

From a *hot* solution in ether or from a mixture of ether and acetone, small, almost colourless, very lustrous crystals were obtained.

Monoclinic-prismatic.

$$a : b : c = 0{,}7958 : 1 : 0{,}8067;$$

$$\beta = 82°9'.$$

Forms observed: $a = \{100\}$ and $m = \{110\}$, well developed and highly lustrous; $c = \{001\}$, also large, but much duller; $r = \{905\}$ and $o = \{111\}$, distinct, but yielding mostly only faint reflections. The angles in the zone of the *b*-axis oscillate within rather wide limits with separate

Fig. 15

individuals. The habit is short-prismatic parallel the *c*-axis or tabular parallel {001}. See Fig. 15.

Angular values:	*Observed:*	*Calculated:*
$a : m = (100) : (110) =$	* 51° 45′	—
$m : o = (110) : (111) =$	* 32 49	—
$m : c = (110) : (001) =$	* 85 9	—
$m : m = (110) : (1\bar{1}0) =$	76 30	76° 30′
$a : r = (100) : (905) =$	26 53½	26 48
$r : c = (905) : (001) =$	55 10½	55 21
$a : c = (100) : (101) =$	82 4	82 9
$c : o = (001) : (111) =$	52 20	52 20

Distinctly cleavable parallel to {001}. The corrosion-figures obtained on {100} with Cassia-oil, were in full agreement with the symmetry mentioned.

On *a* and *c* normal extinction. The plane of the optical axes is {010}; the dispersion is inclined and extraordinarily strong, with: $\varrho > v$.

The apparent axial angle (in Cassia-oil) is about 47½°; one of the axes is seen almost perpendicular to {001}, the first bisectrix almost halving the angle of the edges of (100) and (001) with (010).

Groningen, Laboratory for Inorganic and Physical Chemistry of the University.

Chemistry. — *"On the Crystal-structure of Lead-Iodide"*. By P. TERPSTRA and H. G. K. WESTENBRINK. (Communicated by Prof. F. M. JAEGER).

(Communicated at the meeting of January 30, 1925).

§ 1. On slow evaporation of a hot saturated solution of the salt in water, *lead-iodide*: PbJ_2 crystallizes in very thin hexagon-shaped plates, which in convergent polarized light show, between crossed nicols, the interference-image of a strongly birefracting, optically uniaxial crystal, without circular polarisation and with negative optical character. By means of goniometric measurements with the theodolite-goniometer, it was stated that the bordering of these tables consists in an apparently hexagonal bipyramide, whose faces are, however, so small, that the reflection-images of the collimator-slit are broadened by diffraction to such an extent, that accurate determination of the angles between these narrow facets and the basal plane was not well possible. The mean value of a dozen measurements indicates that the angle looked for must be about 60° 20′, — the axial ratio, therefore, being approximately: 1 : 1,51. This result is in contradiction with NORDENSKJÖLD's measurements [1]), who gives the value of $a : c$ of PbJ_2 as being 1 : 1,2945.

The RÖNTGEN-spectrographic investigation of the salt was started by preparing a radiogram after LAUE's method. The primary pencil of RÖNTGEN-rays was orientated perpendicularly on a very thin hexagonal plate. The RÖNTGEN-image obtained manifests an axis of threefold symmetry and three symmetry-planes intersecting at 120°. *Lead-iodide*, therefore, belongs to one of the following three symmetry-classes: either to the *trigonal-trapezohedral* class (A_3 ; $3L_2$ $o\Pi_3$; oP ; oC), or to the *ditrigonal-scalenohedral* class ($A_3 (= \bar{A}_6)$; $3L_2$; $o\Pi_3$, $3P$; C), or to the *ditrigonal-pyramidal* class (A_3 ; oL_2 ; $o\Pi_3$; $3P$; oC). The principal zones may be readily seen from the Fig. 1, which gives a gnomonic projection of this LAUE-radiogram; with the help of it the indices mentioned could easily be attributed to the numerous diffraction-spots. It appears that the spots $(11\bar{1})$, $(1\bar{1}\bar{1})$, $(\bar{1}11)$, (in MILLER's notation) are in such distance from the spot (111), that the enclosed angles are about 60°; according to the LAUE-radiogram it appears once more as certain, that the axial ratio: $a : c = 1 : 1{,}5$ is most probable.

Secondly a powder-spectogram after DEBIJE-HULL's method was made. The finely powdered PbJ_2 was spread upon a fine silk wire, stretched

[1]) E. NORDENSKJÖLD, Bihang Svenska Kl. Akad., (1874), **2**, Handl. 2.

coaxially in a cylindrical camera. After an exposure of 5 hours (50 Kilo-Volts; about 6 Milli-Amp.) to the rays of a copper-anticathode, a good diffraction-image was obtained; however, some lines in it were conspicuous by their great breadth. In Table I all data with respect to

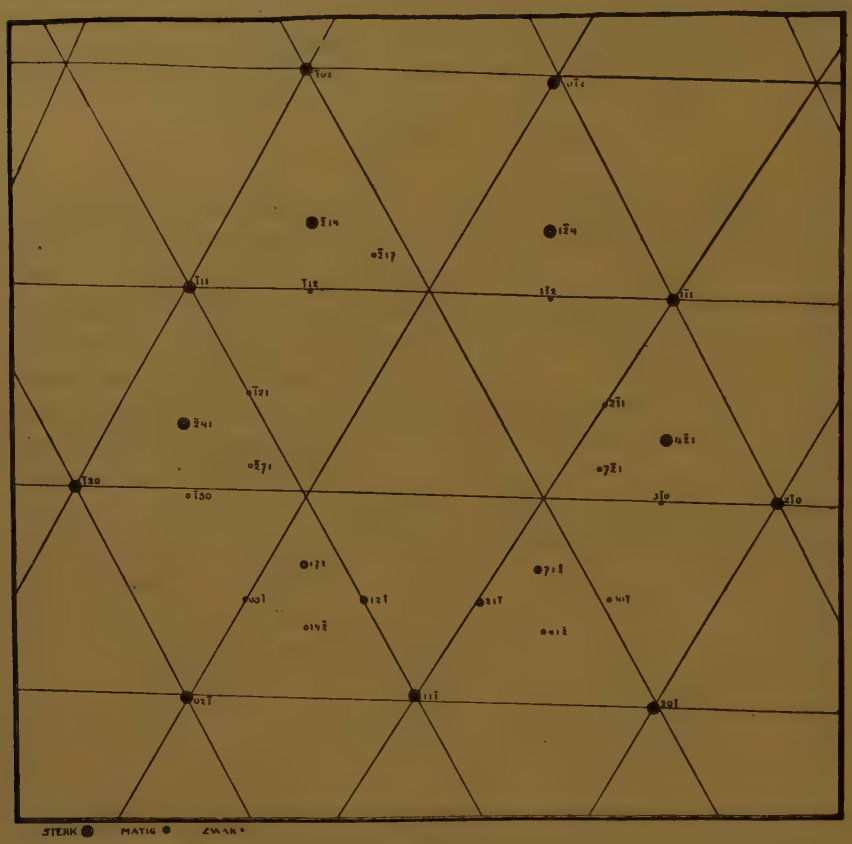


Fig. 1.

these lines are put together; the images No. 3 and No. 5 appeared $\left(\frac{\lambda_\alpha}{\lambda_\beta} \text{ being } = 1{,}107\right)$ to correspond to the β-radiation, the lines No. 4 and No. 6 being the corresponding α-images on the same planes.

The values measured of $sin^2 \frac{\theta}{2}$ for the α-lines may be calculated by the quadratic equation:

$$sin^2 \frac{\vartheta}{2} = 0{,}0375\,(h^2 + hk + k^2) + 0{,}0129\,l^2.$$

The values of $sin^2 \frac{\vartheta}{2}$ calculated by means of this formula are also given in Table I. As $3 \times 0{,}0129$ is approximately equal to 0,0375, there occurs coincidation of a number of lines corresponding to different planes

TABLE I.

The values of x and y in: $\sin^2\frac{\vartheta}{2} = x(h^2+hk+k^2)+yl^2$ are calculated from the values indicated in this Table.

No. of the Lines	Intensities (estimated)	Distance on the film in m.M.	Remarks	$\frac{\vartheta}{2}$	$\sin\frac{\vartheta}{2}$	$\sin^2\frac{\vartheta}{2}$ (measured)	$x=0.0375$, $y=0.0129$ $\sin^2\frac{\vartheta}{2}$ (calculated)	h, k, l (BRAVAIS)	Intensities (calculated) for $u=\frac{1}{6}$	$u=\frac{11}{60}$	$u=\frac{1}{5}$	$u=\frac{13}{60}$
0	8	11.25	Within the central halo	6°31′	0.1135	0.01288	$y=0.0129$	001	28	24	20	16
1	2	16.80	β-line of No. 2	9°34′	0.1662	0.01762	—	—	—	—	—	—
2	2	17.90	hazy broad spot	10°21′	0.1797	0.03240	—	100 (?)	—	—	—	—
3	3	20.34	β-line of No. 4	11°46′	0.2039	—	—	—	—	—	—	—
4	10	22.65		13° 6′	0.2267	0.05153	$x+y=0.0504$; $4y=0.0516$	101 ; 011; 002	21	23	26	28
5	2	27.20	β-line of No. 6	15°44′	0.2711	—	—	—	—	—	—	—
6	8	29.97		17°21′	0.2982	0.08880	$x+4y=0.0891$	102 ; 012	22	22	22	21
7	7	34.55		19°59′	0.3417	0.11696	$3x=0.1125$; $9y=0.1161$	110 ; 001 ;	15	16	16	16
8	3	36.30		21° 1′	0.3587	0.12866	$3x+y=0.1254$	111 ; $2\bar{1}1$	14	12	10	8
9	6	39.48		22°51′	0.3884	0.15085	$x+9y=0.1536$; $4x=0.1500$	103 ; $1\bar{1}3$; 200	12	12	12	12
10	6	41.86		24°14′	0.4104	0.16843	$4x+y=0.1629$; $3x+4y=0.1641$	201 ; $2\bar{2}1$; 112 ; $2\bar{1}2$	6	6	7	8
11	6	46.27		26°47′	0.4506	0.20340	$16y=0.2064$; $4x+4y=0.2016$	004; 202 ; $2\bar{2}2$	8	9	9	10
12	4	49.39		28°35′	0.4785	0.22891	$x+16y=0.2439$; $3x+9y=0.2286$	104 ; $1\bar{1}4$; 113 ; $2\bar{1}3$	7	6	5	3
13	2	54.00		31°15′	0.5188	0.26915	$4x+9y=0.2661$	203 ; $2\bar{2}3$	6	5	5	5
14	5	55.72		32°15′	0.5336	0.28510	$7x+y=0.2754$	211 ; 121	6	6	7	8
15	7	59.31	broad	34°20′	0.5640	0.31809	$7x+4y=0.3141$; $25y=0.3225$; $3x+16y=0.3189$	212 ; 122; 005; 114 ; $2\bar{1}4$			15	16
16	5	62.69		36°17′	0.5918	0.35046	$9x+y=0.3504$; $4x+16y=0.3564$	301 ; $3\bar{3}1$; 204 ; 024			8	7
17	3	65.95		38°10′	0.6180	0.38192	$7x+9y=0.3786$	213 ; 123			8	7
18	1	74.00	very faint	42°50′	0.6799	0.46240	$12x+y=0.4629$; $36y=0.4644$	221 ; $4\bar{2}1$; 006				
19	1	84.50	very faint	48°54′	0.7536	0.56791	$12x+9y=0.5661$	223 ; $4\bar{2}3$				

Some of the equally observed, but very faint lines of the β-radiation, etc., were not mentioned here, because they were of no interest with respect to the theoretical interpretation, the values of $\frac{\vartheta}{2}$ being too uncertain in these cases. If $\lambda_\alpha = 1.540$ Å. E., then: $x=\frac{\lambda^2}{3a^2}=0.00375$; and $y=\frac{\lambda^2}{4c^2}=0.0129$.

From this follows: $a=4.59$ Å. E.; $c=6.78$ Å. E.; and therefore: $a:c=1:1.476$.

of the structure. It is this phenomenon by which the abnormal broadening of a number of lines, as mentioned above, is readily explained. The indices h, k and l in the equation and in the Table are indicated with respect to a system of coordination-axes as drawn in Fig. 2; they are the ordinary indices according to BRAVAIS-MILLER's notation [1]), but in sequence of: $\{h\,k\,i\,l\}$, — the third being left out as superfluous. The numeral coefficients in the above equation have in the hexagonal system the meaning [2]) of: $\frac{\lambda^2}{3a^2}$, resp. $\frac{\lambda^2}{4c^2}$; therefore, it follows from the formula given above:

$$\left(\frac{c}{a}\right)^2=\frac{3}{4}\times\frac{0{,}0375}{0{,}0129},\quad \text{or:}\quad a:c=1:1{,}476.$$

As $\lambda_a=(1{,}537\ A.U.$, the values of a and c, being the dimensions of the space-lattice in these directions, are:

$$a=4{,}59\ A.U.;\quad c=6{,}78\ A.U.$$

§ 2. The values for the specific weight of PbJ_2 given in literature vary between 6,1 and 6,3. For this reason the specific weight of crystals deposited from an aqueous solution was once more determined; the true value at 15° C. was: 6,18. As the volume of the elementary parallelopipedon (Fig. 2) is: $\frac{1}{2}\,a^2c.\sqrt{3}=$ $=\frac{1}{2}\,.\,(4{,}59)^2\,.\,6{,}78$ cubic $A.\,U.=$ $=123{,}8$ cubic $A.\,U.$, a crystal-mass of this volume weighs: $765\,.\,10^{-24}$ grams. The absolute molecular weight of the *iodide* being:

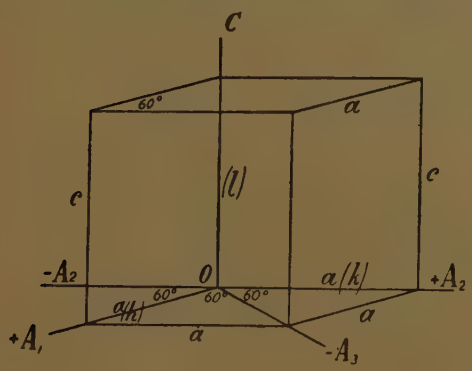


Fig. 2.

[1]) Commonly the indices in the hexagonal system have hitherto been written: $\{hikl\}$. With respect to the RÖNTGEN-spectrographic investigations, however, it seems to be preferable to adopt in future the sequence $\{hkil\}$. In transforming these indices into the MILLERian ones (trigonal axes) $\{pqr\}$, this alteration must be taken into account, so that:

$$p=h-i+l;\ q=k-h+l;\ r=i-k+l;$$

and reversely:

$$h=\frac{p-q}{3}\ ;\ k=\frac{q-r}{3}\ ;\ i=\frac{r-p}{3}\ ;\ \text{and}\ l=\frac{p+q+r}{3}$$

are valuable in this case.

[2]) For the simple hexagonal space-lattice the relation holds:

$$d_{(hkl)}=\frac{}{\sqrt{\tfrac{4}{3}(h^2+hk+k^2)+\frac{a^2}{c^2}l^2}}.$$

But as also : $2d.\sin\theta/2=n.\lambda$, it follows immediately from these two relations, that:

$$\sin^2\theta/2=n^2\left\{\frac{\lambda^2}{3a^2}(h^2+hk+k^2)+\frac{\lambda^2}{4c^2}l^2\right\}.$$

$(207,2 + 253,8) . 1,64 . 10^{-24}$ grams $= 756 . 10^{-24}$ grams, the number of molecules PbI_2 pro elementary cell will be: $\frac{765}{756} = 1,01$, i.e. *one* molecule pro cell.

It may be remarked here that a quadratic equation: $sin^2 \frac{\theta}{2} = 0,0125 (m^2 + mn + n^2) + 0,0129 l^2$ evidently agrees as well with the values of Table I, the coefficient x' here being $\frac{1}{3}$ x of the former. From this equation are calculated: $a' = 7,95$ *A. U.* $= 4,95\sqrt{3}$ *A. U.*, and $c = 6,78$ *A. U.*, each elementary cell now containing 3 molecules of PbI_2. This apparent ambiguity in the choice of the quadratic formula is wholly determined by the arbitrary choice of the axial system with hexagonal crystals, as the axes may be chosen so as to differ from each other by a rotation over 30° round the hexagonal axis. It remains only a question of preference, which one wishes to consider as *a*-axes or as secondary axes. In Fig. 3 the two kinds of elementary parallelopipedic cells are represented in their mutual dependence; their simple relation to each other is easily seen from this figure.

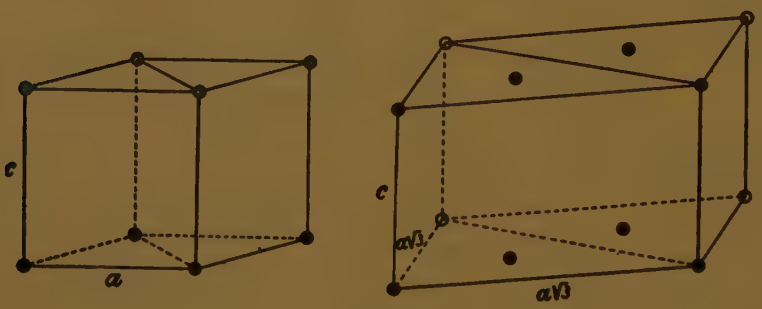


Fig. 3.

§ 3. For the purpose of determining the exact position of these two elementary cells with respect to the hexagonal boundaries of the crystal-plates, two rotational spectrograms *A* and *B* of such a plate were prepared: in the first case the plate was revolved round an axis in the plane of it and parallel to one of the bordering edges; in the second case the axis of rotation of the crystal was also situated in the same plane, but perpendicular to one of the hexagon-edges. In both cases the plane of the crystal-plate was in its mean position, therefore, perpendicular to the incident pencil of RÖNTGEN-rays; from this mean position it was by means of a mechanism, slowly rotated over about 35° to both sides alternately. The diffraction-image was produced on a flat photographic plate, this being placed perpendicularly to the incident beam at a distance of 39 mM.

from the crystal. By this method the socalled "identity-curves" [1]) appear on the film as a system of hyperbolae. Thus, in Fig. 4, giving a reproduction of the rotational spectogram A, the hyperbolae N°. 0, 1, 2, — 1 and — 2, may be easily discerned. From their mutual distances, the socalled

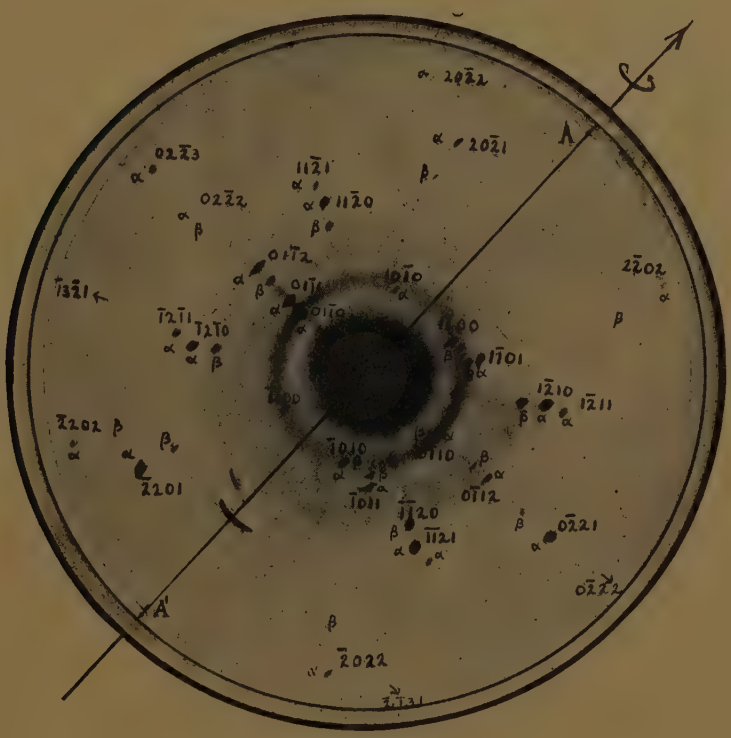


Fig. 4.

"identity-spacings" I in the direction of the axis of rotation can be calculated by means of the formula: $I = \frac{n \cdot \lambda}{\sin \sigma_n}$, in which σ_n is the angle between the direction of the primary beam and the line joining the top of the n^{th} hyperbola in the diffraction-image with the crystal. In Fig. 4 the distance of the top of the 1^{st} hyperbola N°. 0 (horizontal straight line) appears to be 14 mM.; the corresponding distance for the next (2^{nd}) hyperbola being 35 mM.; etc. From this is deduced: $tg\ \sigma_1 = \frac{14}{39}$ and $tg\ \sigma_2 = \frac{35}{39}$; I_1 being thus equal to $\frac{\lambda_\alpha}{\sin \sigma_1} = 4{,}56\ A.U.$; $I_2 = \frac{2.\lambda_\alpha}{\sin \sigma_2} = 4{,}61$ $A.U.$; etc.

[1]) Compare: F. M. JAEGER: *Introduction to the Study of Crystallography*, Chapter V § 18, p. 363, 364. The German authors discern these curves as "Schichtlinien". See: M. POLANYI and K. WEISSENBERG: Zeitschr. f. Physik, **9**, 123, (1922); **10**, 41, (1922); H. W. GONELL and H. MARK, Zeitschr. f. physik. Chemie, **107**, 181, (1923).

On the rotational spectrogram B chiefly both the first and second hyperbolae could be readily discerned: they lay much nearer to each other than those of the spectrogram A, and give: $tg\ \sigma_1 = \frac{8}{40}$, I_1' being $= 7{,}86\ A.U = 4{,}54\sqrt{3}\ A.U$. In this way the true position of both the elementary parallelopipeda mentioned above with respect to the boundaries of the plates becomes immediately clear: they must be situated as indicated in Fig. 5.

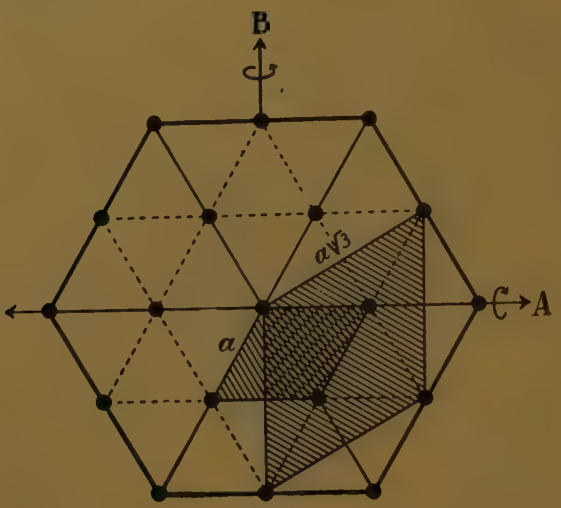


Fig. 5.

Besides about 30 spots for the α-radiation, in Fig. 4 there appear also a number of images corresponding to the β-radiation, which are arranged in the same way on a system of hyperbolae; in connection with the smaller wave-length ($\lambda_\beta = 1{,}388\ A.U.$), these hyperbolae are, as must be expected, nearer to each other than in the case of the α-spots. They can, of course, as well as the α-spots, be used for the purpose of calculation; in the quadratic equation for these β-images the coefficients are $\left(\frac{\lambda_\beta}{\lambda_\alpha}\right)^2$ times *smaller* than those in the quadratic formula used before. Fig. 4 also very distinctly shows the branches of the lemniscate-shaped curves, which, crossing in the centre, have their tops on the axis AA' of rotation. In Table II a review is given of the images on the spectrogram A, corresponding to the α-radiation.

In the second column of this Table the distances are indicated of the different spots and the point, were the primary beam pierces the film (centre). From these numbers it is easily seen, that the spots are distributed on a system of about seven concentric circles; the distance between cristal-plate and film (= 39 mM.) being known, the numbers of the third and fourth columns could be readily calculated.

At the moment that a set of parallel molecular planes of the structure

has such a position, that it "reflects" the X-rays, the perpendicular on this set includes an angle of $\left(90° - \frac{\vartheta}{2}\right)$ with the direction of the primary beam; if a stereographic net of WULFF be made use of, it is easy to construct a stereographic projection of the poles of the "reflecting" planes in their right positions by means of the numbers in the third column of the Table. The plane of projection is chosen in this case perpendicular to the incident pencil of rays. By drawing, in this stereographic projection, the small circles, along which the poles of the planes move to and fro during the rotation of the crystal, the true "mean" position of the poles can in this way be deduced with a relatively great accuracy. In this way the data of the seventh column of Table II were collected; in Fig. 6 a stereographic projection of the poles thus

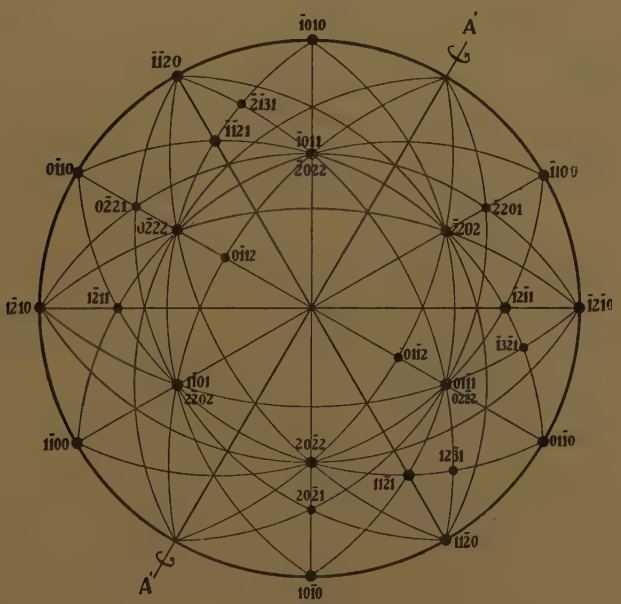


Fig 6.

obtained is reproduced. By combination of the data given in the fourth and seventh column, the valid quadratic equation may be deduced once more; it was found to be: $\sin^2 \frac{\vartheta}{2} = 0{,}0375\ (h^2 + hk + k^2) + 0{,}0131\ l^2$. (Compare also the 5th and 6th column of Table II).

The rotational spectrogram A, therefore, appears to be in very good agreement with the data formerly obtained by analysis of the powder spectrogram.

§ 4. From all this it may concluded with a high degree of probability, that in the case of PbJ_2 we have to do with a trigonal crystal, more especially of rhombohedral symmetry, whose fundamental structure is characterised by a hexagonal translation-group. As a control, the so-

TABLE II. (Rotational Spectrogram A).

N.B. The spots are arranged here in sequence of the identity-hyperbolae.

No. of spots	Distance to centre in mM.	$\frac{\vartheta}{2}$	$\sin^2\frac{\vartheta}{2}$	$\sin^2\frac{\vartheta}{2}$ (calculated)	$x=0.0375$ $y=0.0131$	Indices $\{h\,k\,i\,l\}$	Intensities
0	52.9	26°48′	0.2033	0.2024	$4x+4y$	$\bar{2}202$	—
1	43.0	23°54′	0.1641	0.1631	$4x+y$	$\bar{2}021$	rather large
3	52.9	26°48′	0.2033	0.2024	$4x+4y$	$\bar{2}022$	—
4	78.0	31°43′	0.2765	0.2756	$7x+y$	$\bar{2}\bar{1}31$	—
5	78.0	31°43′	0.2765	0.2756	$7x+y$	$\bar{1}\bar{3}21$	—
7	34.5	20°45′	0.1255	0.1256	$3x+y$	$\bar{1}\bar{2}\bar{1}1$	—
8	31.5	19°28′	0.1111	0.1125	$3x$	$\bar{1}\bar{2}10$	large
9	15.7	10°58′	0.0362	0.0375	x	$\bar{1}100$	—
10	15.7	10°58′	0.0362	0.0375	x	$\bar{1}010$	—
11	19.5	13°17′	0.0528	0.0506	$x+y$	$\bar{1}011$	large
12	31.5	19°28′	0.1111	0.1125	$3x$	$\bar{1}\bar{1}20$	large
13	34.5	20°45′	0.1255	0.1256	$3x+y$	$\bar{1}\bar{1}21$	—
14	72.3	30°50′	0.2626	0.2675	$4x+9y$	$02\bar{2}3$	—
15	52.9	26°48′	0.2033	0.2024	$4x+4y$	$02\bar{2}2$	—
16	27.0	17°21′	0.0890	0.0899	$x+4y$	$01\bar{1}2$	large
17	19.5	13°17′	0.0528	0.0506	$x+y$	$01\bar{1}1$	large
18	15.7	10°58′	0.0362	0.0375	x	$0\bar{1}10$	—
19	2.75	17°36′	0.0914	0.0899	$x+4y$	$0\bar{1}12$	—
20	43.0	23°54′	0.1641	0.1631	$4x+y$	$0\bar{2}21$	rather large
21	52.9	26°48′	0.2033	0.2024	$4x+4y$	$0\bar{2}22$	—
22	78.0	31°43′	0.2765	0.2756	$7x+y$	$12\bar{3}1$	—
23	34.5	20°45′	0.1255	0.1256	$3x+y$	$11\bar{2}1$	—
24	31.5	19°28′	0.1111	0.1125	$3x$	$11\bar{2}0$	large
25	15.7	10°58′	0.0362	0.0375	x	$10\bar{1}0$	—
26	15.7	10°58′	0.0362	0.0375	x	$1\bar{1}00$	—
27	19.5	13°17′	0.0528	0.0506	$x+y$	$1\bar{1}01$	large
28	31.5	19°28′	0.1111	0.1125	$3x$	$1\bar{2}10$	large
29	34.5	20°45′	0.1255	0.1256	$3x+y$	$1\bar{2}11$	—
30	52.9	26°83′	0.2033	0.2024	$4x+y$	$20\bar{2}2$	—
31	43.0	23°54′	0.1641	0.1631	$4x+y$	$20\bar{2}1$	rather large
32	52.9	26°48′	0.2033	0.2024	$4x+4y$	$2\bar{2}02$	—

called "rhombohedron-criterion", applied to the principal planes mentioned in Table I, appears to confirm this conclusion [1]). If also the trigonal-rhombohedral symmetry of the LAUE-radiogram perpendicular to {0001} is taken into account, the number of possible space-groups in this case appears to be limited to only the following fourteen [2]) groups:

$$C_{3V}^1 \ , \ C_{3V}^2 \ , \ C_{3V}^3 \ , \ C_{4V}^4 ;$$
$$D_3^1 \ , \ D_3^2 \ , \ D_3^3 \ , \ D_3^4 \ , \ D_3^5 \ , \ D_3^6 ;$$
$$D_{3d}^1 \ , \ D_{3d}^2 \ , \ D_{3d}^3 \ , \ D_{3d}^4 .$$

But also from these a number can immediately be excluded, because of the condition, that in each elementary cell *one lead-*, and *two iodine-*atoms must find their places; i. e. only simple and twofold equivalent positions must be present in each cell. This condition is only satisfied [3]) by the five groups:

$$C_{3V}^2 \ ; \ D_3^1 \ ; \ D_3^2 \ ; \ D_{3d}^1 \text{ and } D_{3d}^3 .$$

On further consideration it appears that also D_{3d}^1 and C_{3V}^2 can be excluded. For, considering the coordinates of the simple and twofold equivalent positions, these are:

a. for D_{3d}^1; simple equiv. positions: [000]; and $[00\tfrac{1}{2}]$;
twofold equiv. positions:

$$[\tfrac{1}{3}\ \tfrac{2}{3}\ 0] \ ; \ [\tfrac{2}{3}\ \tfrac{1}{3}\ 0] ;$$
$$[\tfrac{1}{3}\ \tfrac{2}{3}\ \tfrac{1}{2}] \ ; \ [\tfrac{2}{3}\ \tfrac{1}{3}\ \tfrac{1}{2}] ;$$
$$[0\ 0\ u] \ ; \ [0\ 0\ \bar{u}] .$$

If, as this must be the case with a substance of the composition: PbI_2, *only* the simple and twofold equivalent positions within each cell be occupied, the symmetry of the whole arrangement must evidently become hexagonal-holohedral in this group, so that it need no longer be considered in this case. For the same reason also C_{3V}^2 can be excluded beforehand, so there only remain the groups: D_3^1, D_3^2 and D_{3d}^3. The simple and twofold equivalent positions in these three groups have the following coordinates respectively:

in D_3^1: a) simple equiv. positions:

$$[0\ 0\ 0] \ ; \ [\tfrac{1}{3}\ \tfrac{2}{3}\ 0] \ ; \ [\tfrac{2}{3}\ \tfrac{1}{3}\ 0] ;$$
$$[0\ 0\ \tfrac{1}{2}] \ ; \ [\tfrac{1}{3}\ \tfrac{2}{3}\ \tfrac{1}{2}] \ ; \ [\tfrac{2}{3}\ \tfrac{1}{3}\ \tfrac{1}{2}] .$$

b) twofold equiv. positions:

$$[0\ 0\ u] \ ; \ [0\ 0\ \bar{u}] ;$$
$$[\tfrac{2}{3}\ \tfrac{1}{3}\ u] \ ; \ [\tfrac{2}{3}\ \tfrac{1}{3}\ \bar{u}] ;$$
$$\text{and: } [\tfrac{1}{3}\ \tfrac{2}{3}\ u] \ ; \ [\tfrac{1}{3}\ \tfrac{2}{3}\ \bar{u}].$$

---

[1]) P. NIGGLI, *Geometrie des Diskontinuums*, p. 481. The expression is:
$$k + l - h \equiv 0 \text{ (mod. 3)}.$$

[2]) For, if trigonal-bipyramidal symmetry were present here, the LAUE-radiogram should manifest hexagonal symmetry, because of the symmetry-centre superimposed upon the symmetry of the crystal itself.

[3]) R. W. G. WYCKOFF, *The Analytical Expression of the Results of the Theory of Space-Groups*, CARNEGIE-Publication, (1922), p. 152—156, 178.

In the same way for D_3^2:

a) simple equiv. positions:

$$[0\,0\,0]\,;\,[0\,0\,\tfrac{1}{2}].$$

b) twofold equiv. positions:

$$[0\,0\,u]\,;\,[0\,0\,\bar{u}]\,;$$
$$\text{and: } [\tfrac{1}{3}\,\tfrac{2}{3}\,u]\,;\,[\tfrac{2}{3}\,\tfrac{1}{3}\,\bar{u}].$$

Finally for D_{3d}^3:

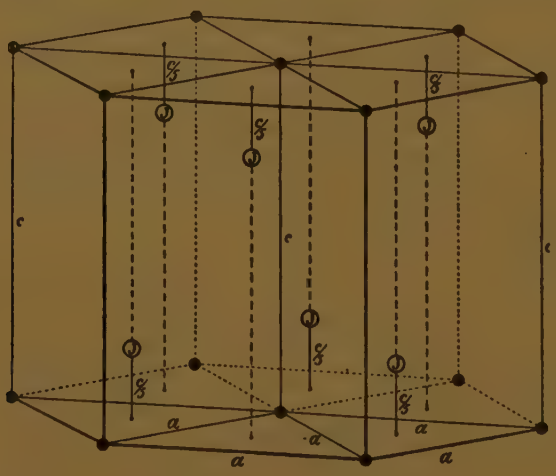
a) simple equiv. posititions:

$$[0\,0\,0]\,;\,[0\,0\,\tfrac{1}{2}].$$

b) twofold equiv. positions:

$$[0\,0\,u]\,;\,[0\,0\,\bar{u}]\,;$$
$$\text{and: } [\tfrac{1}{3}\,\tfrac{2}{3}\,u]\,;\,[\tfrac{2}{3}\,\tfrac{1}{3}\,\bar{u}].$$

Therefore, the coordinates of the simple, resp. twofold equivalent positions are identical for D_3^2 and D_{3d}^3; if *only* these places are occupied and the specific symmetry of the atoms themselves are left out of account, these two cases are identical. The combinations of the positions: [0 0 0] and [0 0 *u*] in D_3^1, or of: [0 0 0] and [0 0 *u*] in D_3^2, resp. $D_{3\,d}^3$ would again lead to holohedral hexagonal symmetry of the structure. It is, therefore, preferable to try the combination of the simple equivalent position: [0 0 0] together with the two twofold equivalent positions: $[\tfrac{1}{3}\,\tfrac{2}{3}\,u]$; $[\tfrac{2}{3}\,\tfrac{1}{3}\,\bar{u}]$. It appears that when *u* be taken $= \tfrac{1}{6}$ (of the *c*-distance), the arrangement thus obtained really quite sufficiently explains



● *Lead-atoms* ◑ *Iodine-atoms*

Fig. 7.

all the characteristic particularities observed in the RÖNTGEN-spectograms. (see Table I and fig. 7).

For it is possible to calculate all the intensities of the diffraction-

images on the powder-spectrograms from the arrangement so adopted, by means of the well-known formula:

$$I = \frac{1 + \cos^2\theta}{2\cos\frac{\theta}{2} \cdot \sin^2\frac{\theta}{2}} \cdot n \cdot |S|^2,$$

in which equation the structural $|S|$ is here:

$$|S| = \left| 82 + 53\left\{ e^{2\pi \cdot i\left(\frac{h}{3} + \frac{2k}{3} + u.l\right)} + e^{2\pi \cdot i\left(\frac{2h}{3} + \frac{k}{3} + (1-v) \cdot l\right)} \right\} \right|.$$

The results obtained in this way for different values of u are put together in Table I. Taking into account the small degree of accuracy reached in estimating visually the intensities of the different lines, it must be affirmed that the value $u = \frac{1}{6}$ gives indeed a close agreement between observed and calculated intensities. A factor of uncertainty is, however, present also in these calculations, as in recent times doubt has arisen about the question, whether the formula mentioned really does express the true relation between I and $|S|$; while it is yet equally uncertain, whether the new proposed formula may be used without objection instead of the old one in cases like this [1]).

In the rotational spectrograms the relation between the density of the molecular planes and the intensities of the interference-images is much more complicated even than in the powder-spectrograms. For this reason they are also less suited for the comparison of the observed and calculated intensities yet. It is possible to give a satisfactory explanation of the typical trigonal character of the rotational spectrogram A for instance, by means of the specific properties of the structural factor:

$$|S| = \left| 82 + 53\left\{ e^{2\pi \cdot i\left(\frac{h}{3} + \frac{2k}{3} + \frac{l}{5}\right)} + e^{2\pi \cdot i\left(\frac{2h}{3} + \frac{k}{3} + \frac{4l}{5}\right)} \right\} \right|;$$

for it may be easily seen that this factor gets a great value, for instance for the planes: $(01\bar{1}1)$, $(1\bar{1}01)$ and $(\bar{1}011)$, while it is only small for the planes: $(\bar{1}101)$, $(101\bar{1})$ and $(0\bar{1}11)$; etc.

The structure of Fig. 7 is analogous to that attributed by AMINOFF [2]) to the ditrigonal-scalenohedral minerals: *Pyrochroite*: $Mn(OH)_2$ and *Brucite*: $Mg(OH)_2$, and also to the structure derived by BOZORTH [3]) of *Cadmium-iodide* [4]): CdI_2.

§ 6. Besides the ROENTGENograms discussed above (LAUE-, powder- and rotational spectrograms), also some spectra on {0001} were investigated

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[1]) Compare: P. P. EWALD, Phys. Zeitschr., **26**, 29, (1925); H. OTT, Zeitschr. f. Kryst., **61**, 515, (1925); H. MARK, *Die Naturwissenschaften*, **13**, 1045, (1925).

[2]) G. AMINOFF, Geol. Förh. Stockholm, **41**, 407, (1919); Zeitschr. f. Kryst., **56**, 506, (1921).

[3]) R. M. BOZORTH, Journ. Amer. Chem. Soc., **44**, 2232, (1922).

[4]) It is remarkable that from the crystallographic measurements of CdI_2 an axial ratio $a:c$ is deduced, which appears to be very nearly twice that of PbI_2; in connection with the ROENTGEN-spectroscopic determinations that value must even be divided by 2, so that the agreement between the two substances becomes yet closer.

after BRAGG's method. In one of these experiments some small hexagon-shaped crystals of PbI_2 were stuck on a cleavage-plate of *calcite*; in this way there appear, besides the lines of PbI_2, on the film, simultaneously those of *calcite*. It appeared that the α-spectrum of the *second* order on {0001} of PbI_2 is very close to the *first* order spectrum for the β-radiation on the same plane of *calcite*. By means of this film the spacing of the equivalent molecular planes parallel to {0001}, — i. e. the value of c, — was determined to be: 6,862 *A. U.*. From this value for c and the specific weight: 6,18 of PbI_2, and assuming that only *one* molecule of the salt be present in the elementary cell, the values for a and c can be calculated as: $a = 4{,}54$ *A. U.*; $c = 6{,}86$ *A. U.*

In agreement with what could be expected according to the structure derived above, the „reflections" of the 2nd and 3rd order on this film appeared to be much fainter than those of the 1st order. It must be remarked, however, that not too great a demonstrative value may be attributed to the observed relation of the intensities, because an accurate mounting of these crystals is rather difficult, as a consequence of their very minute size.

Although both limiting rhombohedra $\{01\bar{1}1\}$ and $\{10\bar{1}1\}$ do really much differ with respect to the reticular density of their planes, it seems highly remarkable, that the crystal-plates appear always to be bordered by an apparently complete hexagonal bipyramide. They possess evidently both comparable velocities of growth, notwithstanding their different molecular structure, which velocities seem, besides the minimal velocity of growth of the basis {0001} itself, to be rather minimal values among all others.

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Mathematics. — *"Ueber stetige Abbildungen einer Kugelfläche"*. Von L. VIETORIS, Wien, dzt. Amsterdam. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of February 27, 1926).

Wir wollen im Folgenden nebst Anwendungen und einer Verschärfung den Satz beweisen:

(1) Eine eindeutige stetige Abbildung einer Kugelfläche K auf eine Kurve[1]) $\mathfrak{K}$, bei der die Urbilder jedes Punktes von $\mathfrak{K}$ ein Kontinuum[2]) bilden, besteht dann und nur dann, wenn die Kurve ein Baum[3]) ist.

Ein kompakter metrischer Raum K sei auf einen anderen $\mathfrak{K}$ eindeutig und stetig abgebildet. Bezeichnen wir dann die Menge aller Urbilder eines Punktes y von $\mathfrak{K}$ als die Urbildmenge Y von y, so sind die zu verschiedenen Punkten y_1, y_2 von $\mathfrak{K}$ gehörigen Urbildmengen zu einander fremd. Für die Beziehung zwischen y und Y gilt[4]):

Ist $y_1, y_2, y_3 \ldots$ eine konvergente Folge in $\mathfrak{K}$ mit dem Limes y_0, dann ist $\overline{lim}$ sup $Y_i \subseteq Y_0$.[5])

Diese Eigenschaft entspricht genau der Halbstetigkeit nach oben in der Lehre von den reellen Funktionen. Wir nennen daher die zu unserer Abbildung gehörige Zerlegung des Urbildraumes im Anschluss an R. L. MOORE[6]) und P. ALEXANDROFF eine *oberhalb stetige Zerlegung* von K.

Definiert man für die Elemente Y der Zerlegung in geeigneter Weise

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[1]) Kurve = eindimensionales Kontinuum.

[2]) Diese Abbildungen verdienen, auch losgelöst von Satz (1), als natürliche Verallgemeinerungen der topologischen Abbildungen und wegen der folgenden Sätze (2), (3), (3*a*) Beachtung.

[3]) Baum = im Kleinen zusammenhängende abgeschlossene kompakte Kurve, die keinen topologischen Kreis enthält. Vgl. K. MENGER, „Ueber reguläre Baumkurven", Math. Ann. 96, wo auch die ältere Literatur zitiert ist.

[4]) P. ALEXANDROFF, „Ueber stetige Abbildungen kompakter Räume", These Proceedings 28, S. 997—999, sowie Math. Ann. 96.

[5]) $\overline{lim}$ sup = oberer abgeschlossener Limes (HAUSDORFF) = obere Näherungsgrenze (HAHN, Reelle Funktionen).

[6]) Proc. Nat. Ac. 10, p. 356—360 und Am. Trans. 27, p. 416—428. MOORE nennt diese Zerlegungen „upper semicontinuous collections", ALEXANDROFF im Hinblick auf die Beziehung zu den stetigen Funktionen bloss „stetige Zerlegungen". Wir werden hier auch mit Zerlegungen zu tun haben, die ausserdem in analogem Sinn unterhalb stetig sind, und diese dann stetig schlechtweg nennen.

Umgebungen [7]) oder Abstände [8]), so erhält man einen topologischen, bezw. metrischen Raum $\mathfrak{R}^*$, der zu $\mathfrak{R}$ homöomorph ist und *Zerlegungsraum* [7]) heisst.

Indem wir $\mathfrak{R}$ mit $\mathfrak{R}^*$ identifizieren, können wir Satz (1) auch folgendermassen aussprechen:

(1′) Eine Kugelfläche ist dann und nur dann in eine vorgegebene eindimensionale Menge von Kontinuen oberhalb stetig zerlegbar, wenn diese ein Baum ist.

Z.B. ist die Zerlegung der Kugelfläche nach Parallelkreisen eine oberhalb (und unterhalb) stetige Zerlegung derselben in eine Strecke. Wollen wir die Kugelfläche, bezw. die durch einen uneigentlichen Punkt ergänzte Ebene, oberhalb stetig in Form eines T zerlegen, so nehmen wir in der Ebene ein durch einen Querschnitt Q in zwei Teilrechtecke zerfallendes Rechteck L an (Fig. 1). Sind u der unendlich ferne Punkt der Ebene, a und b die Mitten der Teilrechtecke, so definieren wir als Elemente der Zerlegung die Punkte u, a, b, ferner $L + Q$ und drei kontinuierliche Scharen von Rechtecken, von denen die eine sich von L aus gegen u, die beiden anderen von den beiden Teilrechtecken aus gegen a und b zusammenziehen. Der zugehörige Zerlegungsraum ist ein T-förmiger Baum, in dem $L + Q$ der Punkt dritter Ordnung, u, a, b die Punkte erster Ordnung sind.

Fig. 1.

Wir wollen zuerst zeigen, dass jeder eindimensionale oberhalb stetige Zerlegungsraum einer Kugelfläche, dessen Elemente Kontinua sind, ein Baum ist.

Wir tun dies indem wir feststellen, dass die Kugelfläche die folgenden für die Bäume unter den eindimensionaler Mengen charakteristischen Eigenschaften der *Kompaktheit in sich, des Zusammenhangs, des Zusammenhangs im Kleinen* und des *einfachen Zusammenhanges hat*, und zeigen, dass *alle diese Eigenschaften gegenüber den eindeutigen stetigen Abbildungen, in denen jedes Bild ein Kontinuum von Urbildern hat, invariant* sind.

Die Kompaktheit in sich, [9]) der Zusammenhang [9]) und der Zusammenhang im Kleinen im Verein mit der Kompaktheit in sich [10]) sind bekannt-

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[7]) ALEXANDROFF l. c.

[8]) R. L. MOORE l. c.

[9]) HAUSDORFF, Grundzüge, S. 363, 364.

[10]) Letzteres ergibt sich unmittelbar aus der Charakterisierung der stetigen Bilder der Strecke von HAHN, Sitz. Ak. Wien 123 (Dez. 1914) und MAZURKIEWICZ, Fund. Math. I, S. 191.

lich schon gegenüber allen eindeutigen stetigen Abbildungen invariant. Es bleibt also nur mehr bezüglich des einfachen Zusammenhangs etwas zu beweisen.

Dazu zeigen wir zuerst:

(2) Ist ein kompakter Raum K auf einen andern $\mathfrak{R}$ eindeutig und stetig so abgebildet, dass die Menge aller Punkte von K, deren Bild in einen Punkt y von $\mathfrak{R}$ fällt, für jeden Punkt y ein Kontinuum ist, dann haben K und $\mathfrak{R}$ gleich viel Komponenten,

Aus der Invarianz des Zusammenhanges gegenüber eindeutigen stetigen Abbildungen folgt unmittelbar schon, dass $\mathfrak{R}$ nicht mehr Komponenten als K haben kann. Um (2) zu zeigen genügt es, zu beweisen: Ist $\mathfrak{M}$ ein Teilkontinuum von $\mathfrak{R}$, so ist die Menge M aller Urbilder aller Punkte von $\mathfrak{M}$ ein Kontinuum. Angenommen, M zerfalle in zwei nicht leere fremde abgeschlossene Teile M_1, M_2. Da die Menge Y aller Urbilder jedes Punktes y von $\mathfrak{R}$ ein Kontinuum ist, kann sie nur entweder ganz in M_1 oder ganz in M_2 liegen; $\mathfrak{M}$ zerfällt also in zwei nicht leere fremde Mengen $\mathfrak{M}_1$, $\mathfrak{M}_2$, von denen $\mathfrak{M}_1$ jene Punkte y enthält, für welche Y in M_1 liegt, $\mathfrak{M}_2$ jene, für die Y in M_2 liegt. $\mathfrak{M}_1$ und $\mathfrak{M}_2$ sind als Bilder von M_1 und M_2 abgeschlossen entgegen der Annahme, dass $\mathfrak{M}$ ein Kontinuum ist. Wir sagen nun *per definitionem*:

Ein Kontinuum K habe n Henkel, wenn der Durchschnitt irgend zweier supplementärer[11]) Teilkontinua höchstenfalls[12]) $n+1$ Komponenten hat.[13])

Wir zeigen nun:

(3) Wird ein n-henkeliges kompaktes Kontinuum K eindeutig und stetig so auf eine Menge $\mathfrak{R}$ abgebildet, dass die Menge Y der Urbilder jedes Punktes y von $\mathfrak{R}$ ein Kontinuum ist, dann hat $\mathfrak{R}$ höchstens n Henkel.[14])

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[11]) Supplementäre Teile = Teile, deren Summe das Ganze ist, wobei Fremdheit der Teile (im Gegensatz zu komplementär) nicht beinhaltet ist.

[12]) „Höchstenfalls" bedeute im Gegensatz zu „höchstens", dass n tatsächlich erreicht wird.

[13]) Durch Verwendung der Relativumgebungen überträgt sich dieser absolut topologisch invariante Begriff auf beliebige zusammenhängende Mengen. Das Verhältnis zwischen ihm und der *Basis der Zyklosis* BROUWERs (Math. Ann. 72, S. 422—425) soll in einer späteren Abhandlung behandelt werden.

[14]) Es gilt auch der entsprechende Satz bezüglich der Basis der Zyklosis:

(3a) Hat ein kompaktes Kontinuum K eine n-fache Basis der Zyklosis und ist $\mathfrak{R}$ der art eindeutiges stetiges Bild von K, dass die Menge Y der Urbilder jedes Punktes y von $\mathfrak{R}$ ein Kontinuum ist, dann hat auch $\mathfrak{R}$ eine n-fache Basis der Zyklosis.

Der mir samt dem Satz selbst von Prof. BROUWER im wesentlichen mündlich mitgeteilte Beweis verläuft auf folgende Weise:

Ist der kompakte metrische Raum K auf $\mathfrak{R}$ eindeutig stetig abgebildet, so gibt es zu jedem ε mit ε gegen Null konvergierende ε' und ε'', sodass zu irgend zwei Punkten y_1, y_2 von $\mathfrak{R}$, deren Abstand $< \varepsilon$ ist, ein Punkt $y_{1,2}$ in $\mathfrak{R}$ existiert, dessen Urbildmenge $Y_{1,2}$ von den Urbild-

Wir wissen schon, dass $\mathfrak{R}$ ein Kontinuum ist. Nun seien $\mathfrak{R}_1$ und $\mathfrak{R}_2$ zwei supplementäre Teilkontinua von $\mathfrak{R}$. Ihr Durchschnitt heisse $\mathfrak{S}$. Ferner möge die Menge aller Urbilder aller Punkte von $\mathfrak{R}_1$ mit K_1, die analogen Bildungen zu $\mathfrak{R}_2$ und $\mathfrak{S}$ mit K_2 und S bezeichnet werden. Wegen der Eindeutigkeit der Abbildung gilt $K_1, K_2 = S$. Nun sind, weil $\mathfrak{R}_1$ und $\mathfrak{R}_2$ Kontinua sind, nach (2) auch K_1 und K_2 Kontinua. Weil K nur n Henkel hat, hat S höchstens $n+1$ Komponenten, also nach (2) auch $\mathfrak{S}$.

Man überzeugt sich leicht von der Giltigkeit des Satzes:

(4) *Die Kugelfläche hat keinen Henkel.*

(5) Eine im Kleinen zusammenhängende kompakte abgeschlossene Kurve ist dann und nur dann ein Baum, wenn sie keinen Henkel hat.

Nach MENGER [15]) sind die Bäume unter den im Kleinen zusammenhängenden Kontinuen dadurch gekennzeichnet, dass in ihnen irgend zwei Teilkontinua einen leeren oder einen zusammenhängenden Durchschnitt haben. Daher ist zunächst jeder Baum nullhenkelig. Nun enthalte umgekehrt eine im Kleinen zusammenhängende Kurve K einen Kreis L. a sei ein Punkt von L, U eine (Relativ-)Umgebung von a, deren Begrenzung 0-dimensional ist und welche irgend einen Punkt b von L weder als Punkt noch als Häufungspunkt hat. B_1 bezeichne jene Komponente von $K-U$, welche b enthält, während A_1 aus $\overline{U}$ und allen übrigen Komponenten von $K-\overline{U}$ zusammengesetzt sei. Wegen des Zusammenhangs im Kleinen von K sind A_1 und B_1 abgeschlossen. Nun sei A_2 jene Komponente von A_1, welche a enthält, B_2 die Vereinigung von B_1 mit allen übrigen Komponenten von A_1. Dann sind A_2 und B_2 supplementäre Teilkontinua von K, deren Durchschnitt ein Teil der Begrenzung von U, also 0-dimensional ist, aber mindestens zwei Punkte (Punkte von L) enthält.

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mengen Y_1 und Y_2 (Minimal)-Entfernungen $< \varepsilon''$ hat und für den ausserdem $\overline{y_1 y_{1,2}} < \varepsilon'$, $\overline{y_2 y_{1,2}} < \varepsilon'$ gilt.

Ist nun irgend eine ε-Kette $\mathfrak{Z}$ in $\mathfrak{R}$ vorgelegt, so erhalten wir nach dem vorigen Absatz durch geeignete Zwischenschaltung von höchstens je einem Punkt zwischen zwei benachbarte Punkte eine ε'-Kette $\mathfrak{Z}_0$, der im Zerlegungsraum eine Kette $\mathfrak{Z}^*$ von Mengen Y entspricht, in welcher irgend zwei benachbarte Mengen Y voneinander einen Minimalabstand $< \varepsilon''$ haben. Da jedes Y ein Kontinuum ist, kann ich — durch Einfügen von hinreichend vielen Punkten in jedem Y von $\mathfrak{Z}^*$ — eine ε''-Kette Z^0 konstruieren, deren Bild $\mathfrak{Z}_0$ ist. Es gibt nun ein mit ε'', mithin mit ε gegen Null konvergierendes $\varepsilon^{(3)}$, und n Fundamentalketten $Z_1, \ldots, Z_n$ in K, auf welche sich eine beliebige ε''-Kette in K mittels $\varepsilon^{(3)}$-Abänderungen in K reduzieren lässt. Seien $\mathfrak{Z}_1, \ldots, \mathfrak{Z}_n$ die Bilder von $Z_1, \ldots, Z_n$ in $\mathfrak{R}$; $\varepsilon^{(4)}$ eine solche mit $\varepsilon^{(3)}$, also mit ε gegen Null konvergierende Grösse, dass je zwei Punkten von K, deren Abstand $< \varepsilon^{(3)}$ ist, immer zwei Punkte in $\mathfrak{R}$ entsprechen, deren Abstand $< \varepsilon^{(4)}$ ist; $\varepsilon^{(5)}$ die grössere der beiden Zahlen $\varepsilon^{(4)}$ und ε'. Alsdann lässt sich nach dem obigen eine beliebige ε-Kette in $\mathfrak{R}$ mittels $\varepsilon^{(5)}$-Abänderungen in $\mathfrak{R}$ auf $\mathfrak{Z}_1, \ldots, \mathfrak{Z}_n$ als Fundamentalketten reduzieren, sodass $\mathfrak{R}$ in der Tat eine n-fache Basis der Zyklosis besitzt.

[15]) l.c.

Mit (3), (4), (5) haben wir die eine Hälfte von (1) gezeigt. [16]).

Nun sei ein beliebiger Baum $\mathfrak{R}$ *vorgelegt.* Wir wollen eine oberhalb stetige Zerlegung einer Kugelfläche K in Kontinua angeben, deren Zerlegungsraum zu $\mathfrak{R}$ homöomorph ist.

Sind $a_1, a_2, \ldots a_m$ irgend welche Punkte von $\mathfrak{R}$ mit den Ordnungen $p_1, p_2, \ldots p_m$, so gibt es auf K m zu einander fremde Kontinua (die insbesondere als Kurven angenommen werden können) $A_1, A_2, \ldots A_m$ von denen jedes A_i in K p_i Gebiete bestimmt derart, dass die von $\{p_1, p_2, \ldots p_m\}$ in $\mathfrak{R}$ bestimmten Gebiete $\mathfrak{G}_j$ den von $A_1 + A_2 + \ldots + A_m$ in K bestimmten Gebieten G_j so zugeordnet sind, dass G_j von $A_{j_1} + A_{j_2} + \ldots + A_{jk}$ genau dann begrenzt wird, wenn $\mathfrak{G}_j$ von $\{a_{j_1}\ a_{j_2}, \ldots a_{jk}\}$ begrenzt wird.

Wir wählen nämlich ein p_1-fach zusammenhängendes Kontinuum A_1 auf K beliebig und ordnen den von a_1 in R bestimmten [17]) p_1 Gebieten $\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3, \ldots$. irgendwie die von A_1 auf K bestimmten Gebiete als $G_1, G_2, G_3, \ldots$. zu. Liegt dann a_2 in $\mathfrak{G}_1$, so wählen wir ein p_2-fach zusammenhängendes Kontinuum A_2 in G_1 und ordnen dem Zwischengebiet zwischen A_1 und A_2 das auf $\mathfrak{R}$ von a_1 und a_2 bestimmte Zwischengebiet, allen übrigen von A_2 in G_1 bestimmten Gebieten beliebig die übrigen von a_2 in $\mathfrak{G}_1$ bestimmten Gebiete zu. Nun sei $n \leqq m$ und für alle $i < n$ jedem Punkt a_i ein Kontinuum $A_i \subset K$ zugeordnet, welches die Ordnung des Punktes a_i als Zusammenhangszahl hat, sodass ausserdem $A_i\, A_k = O$ ist für $i \neq k$, während die durch $\{a_1, a_2, \ldots. a_n\}$ bestimmten Gebiete $\mathfrak{G}_1, \mathfrak{G}_2, \ldots. \mathfrak{G}_s$ den in gleicher Anzahl vorhandenen von $A_1 + A_2 + \ldots. + A_n$ auf K bestimmten Gebieten in der Weise entsprechen, dass G_i von $A_{i_1} + A_{i_2} + \ldots. + A_{i_r}$ $(i_1, i_2, \ldots. i_r < n)$ genau dann begrenzt wird, wenn $\mathfrak{G}_i$ von $\{a_{i_1}, a_{i_2}, \ldots. a_{ir}\}$ begrenzt ist. All das ist für $n \leqq 2$ erfüllt. Nun liege a_n in $\mathfrak{G}_i$. Die Komponenten C_l von $K - G_i$ setzen sich aus $A_1, A_2, \ldots. A_{n-1}$ und $G_1, G_2, \ldots. G_{i-1}, G_{i+1}, \ldots G_s$ ebenso zusammen, wie die Komponenten von $\mathfrak{R} - \mathfrak{G}_i$ aus den entsprechenden Elementen in $\mathfrak{R}$. Die C_l sind einfach zusammenhängend; denn die Komplementärmenge in K ist $G_i + \sum_{k \neq l} C_k$, d. h. ein Gebiet. Wenn wir etwa C_1 und C_2 in G_i durch einen Querschnitt Q_{12} verbinden, so ist $C_1 + Q_{12} + C_2$ immer noch einfach zusammenhängend. Allgemein können wir in G_i der Reihe nach endlich viele Querschnitte so ziehen, dass durch dieselben die Komponenten C_l zu endlich vielen mit einander fremden einfach zusammenhängenden Kontinuen verbunden werden, von denen jedes zwei Komponenten C_k und C_l dann und nur

16) Da, wie ich in einer späteren Arbeit zeigen werde, ein kompaktes im Kleinen zusammenhängendes Kontinuum dann und nur dann eine 0-fache Basis der Zyklosis hat, wenn es keinen Henkel hat, gilt auch:

(5a) Eine im Kleinen zusammenhängende kompakte abgeschlossene Kurve ist dann und nur dann ein Baum, wenn sie eine 0-fache Basis der Zyklosis hat.

Das ergibt einen anderen Beweis der ersten Hälfte von (1).

17) „Bestimmt" im Sinne BROUWERs (Math. Ann. 69, S. 169—175).

dann enthält, wenn die entsprechenden Komponenten $\mathfrak{C}_k$ und $\mathfrak{C}_l$ in $\mathfrak{R}$ in derselben Komponenten von $\mathfrak{R}-\{a_n\}$ liegen. Wir erhalten so höchstens p_n einfach zusammenhängende zu einander fremde Kontinua auf der Kugel, welche wir durch ein p_n-fach zusammenhängendes Kontinuum A_n von einander trennen. Ordnen wir nun noch die von A_n in G_i bestimmten Gebiete den in $\mathfrak{G}_i$ von a_n bestimmten Gebieten geeignet zu, so ist der für n vorausgesetzte Zustand für $(n+1)$ hergestellt.

Wir nennen die Menge der Kontinuen $A_1, A_2, \ldots A_m$ zur Punktmenge $\{a_1, a_2, \ldots a_m\}$ homolog, ebenso die entsprechenden Gebietsmengen.

Nun teilen wir, um zur Konstruktion der behaupteten Zerlegung der Kugel zu kommen, den Baum $\mathfrak{R}$ in endlich viele Teilbäume $\mathfrak{R}_1^1, \mathfrak{R}_2^1, \ldots \mathfrak{R}_s^1$ von einem Durchmesser $< \frac{1}{2}$,[18]) von denen je zwei höchstens einen Punkt gemein haben, und ordnen der Menge aller Punkte $a_1, a_2, \ldots a_{m_1}$, welche als Durchschnitte zweier Teilbäume auftreten ein homologes System $A_1, A_2, \ldots A_{m_1}$ auf der Kugel zu. Indem wir die von $A_1+A_2+\ldots+A_{m_1}$ auf K bestimmten Gebiete und die Mengen A_j in derselben Weise zusammenfassen, wie die homologen Gebiete in $\mathfrak{R}$ mit den Punkten a_j zu den Kontinuen $\mathfrak{R}_1^1, \mathfrak{R}_2^1, \ldots \mathfrak{R}_s^1$ zusammengefasst sind, erhalten wir auf K die Kontinua $K_1^1, K_2^1, \ldots K_s^1$, die wir zu $\mathfrak{R}_1^1, \mathfrak{R}_2^1, \ldots \mathfrak{R}_s^1$ homolog nennen. Das sei der erste Schritt. Beim n^{ten} Schritt teilen wir alle Teilkontinua $\mathfrak{K}_i^{n-1}$ des $(n-1)^{\text{ten}}$ Schrittes, die alle einen Durchmesser $< \frac{1}{2^{n-1}}$ haben, in Teilkontinua $\mathfrak{K}_i^n$ mit Durchmessern $< \frac{1}{2n}$ und wählen für die hinzukommenden Teilungspunkte $a_{m_{n-1}+1}$, $a_{m_{n-1}+2}, \ldots a_{m_n}$ die Kontinua $A_{m_{n-1}+1}$, $A_{m_{n-1}+2}, \ldots A_{m_n}$ so, dass die ergänzten Systeme der a_i und A_i wieder homolog sind.

Für einen beliebigen Punkt y von $\mathfrak{R}$ ist nun die Vereinigung aller ihn enthaltenden Teilbäume des n^{ten} Schrittes eine Umgebung $\mathfrak{U}_n^y$ von einem Durchmesser $< \frac{1}{2^{n-1}}$. Diesen $\mathfrak{U}_1^y \supseteq \mathfrak{U}_2^y \supseteq \ldots$ entsprechen auf K die homologen Kontinua $U_1^y \supseteq U_2^y \supseteq \ldots$ Wir definieren nun als *das dem Punkt y von $\mathfrak{R}$ zugeordnete Element* Y^* der zu konstruierenden oberhalb stetigen Zerlegung $\mathfrak{R}^*$ von K den Durchschnitt $\prod_{n=1}^{\infty} U_n^y = Y^*$. Y^* ist ein Kontinuum. Ist y ein Punkt a_i, so ist das zu a_i gehörige $A_i^* \supseteq A_i$.

Da zu irgend zwei Punkten y_1, y_2 von $\mathfrak{R}$ fremde Umgebungen $\mathfrak{U}^{y_1}, \mathfrak{U}^{y_2}$ existieren, sind die Mengen Y_1^*, Y_2^* zu einander fremd. Es liegt auch jeder Punkt x von K auf einem Y^*. Denn x liegt für jedes n in einer der Mengen U_n des n^{ten} Schrittes. Die entsprechenden Mengen $\mathfrak{U}_n$ haben wegen der Kompaktheit von $\mathfrak{R}$ einen nicht leeren Durchschnitt, der auch nicht mehr als einen Punkt y enthalten kann, weil die Durchmesser der $\mathfrak{U}_n$ gegen 0 konvergieren.

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[18]) MENGER l.c.

Definieren wir nun, dass *zwei Punkte y von* $\mathfrak{R}$ *und x von K einander in der Abbildung* $y = f(x)$ *entsprechen sollen, wenn x auf* Y^* *liegt*, so ist f eindeutig. Um die Oberhalb-Stetigkeit der Zerlegung $\mathfrak{R}^*$ nachzuweisen, haben wir noch zu zeigen, dass $f(x)$ stetig ist. Es sei y_0 ein Punkt von $\mathfrak{R}$, x_0 ein Punkt von Y_0^*. Die $\mathfrak{U}_n^{y_0}$ entsprechende Menge $U_n^{y_0}$ enthält Y_0^* und damit x_0 im Innern und zwar liegt ihr Bild gemäss f in $\mathfrak{U}_n^{y_0}$, worin die Stetigkeit von f ja besteht.

Die Y^* müssen bei dieser Konstruktion nicht lauter Kurven, bezw. Punkte sein. Dies können wir aber erreichen, indem wir bei der Wahl der A_i so vorgehen: Wir nehmen in K eine überall dichte abzählbare Menge $b_1, b_2, \ldots$ an. Ist dann A_n in dem Gebiet G_i zu wählen, so tun wir dies wie bisher, indem wir ausserdem A_n als Kurve wählen, die von dem ersten in G_i liegenden Punkt b_j einen Minimalabstand $< \frac{1}{n}$ hat. Dadurch liegt dann die Menge $\sum_{i=1}^{\infty} A_i$ in K dicht und es kann kein Y^* innere Punkte enthalten, ist also eine Kurve.

Wegen (2) bestimmen y in $\mathfrak{R}$ und Y^* in K gleich viel Gebiete, d.h. *Y^* hat die Ordnung von y als Zusammenhangszahl.* Entsprechend den Verhältnissen auf Bäumen gilt also für eindimensionale oberhalb stetige Zerlegungen der Kugelfläche in lauter Kontinua: *Die Zerlegung hat nur abzählbar viele mehr als zweifach zusammenhängende Elemente, mindestens zwei einfach zusammenhängende Elemente und überall dicht von der Mächtigkeit* $\mathfrak{c}$ *zweifach zusammenhängende Elemente.* Die Elemente der Zerlegung sind insbesondere nicht alle untereinander homöomorph.

Nach R. L. MOORE (a. a. O.) liefert jede oberhalb stetige Zerlegung der Kugelfläche in lauter einfach zusammenhängende Kontinua einen zur Kugelfläche homöomorphen Zerlegungsraum und umgekehrt. Insbesondere erhalten wir eine zur Kugel homöomorphe oberhalb stetige Zerlegung $\mathfrak{R}^*$ der Kugel K, wenn wir in K eine abzählbare Menge von paarweise fremden einfach zusammenhängenden Kontinuen, deren Durchmesser gegen 0 konvergieren, annehmen und als Elemente der Zerlegung einerseits diese Kontinua, anderseits alle darin nicht liegenden Punkte von K ansehen. Lassen wir aus K alle Punkte, die im Inneren irgend eines dieser Kontinua liegen, weg, so ist der übrig bleibende Teil K_1 von K immer noch oberhalb stetig zerlegt, nämlich in die Ränder der Elemente von $\mathfrak{R}^*$. Der zugehörige Zerlegungsraum $\mathfrak{R}_1^*$ ist zu $\mathfrak{R}$ homöomorph, also wieder topologisch eine Kugelfläche.

Auf diese Weise können wir auf der Kugel Kurven K_1 angeben, die oberhalb stetig in Kurven und Punkte so zerlegt sind, dass der Zerlegungsraum eine Kugel ist, d. h. die eindeutig und stetig auf eine Kugel K_2 so abgebildet werden können, dass jedem Punkt der Kugel als Menge von Urbildern eine Kurve oder ein Punkt entspricht. Bilden wir nun K_2 eindeutig stetig so auf einen beliebig vorgegebenen Baum $\mathfrak{R}_2$ ab, dass jedem Punkt x von $\mathfrak{R}_2$ in K_2 ein Kontinuum von Urbildern entspricht, so ist die dadurch vermittelte Abbildung von K_1 auf $\mathfrak{R}_2$ wieder eindeutig

stetig und ordnet jedem Punkt x von $\mathfrak{R}_2$ ein Kontinuum von Urbildern in K_1 zu. Dass diese Kontinua auch sämtlich aus mehr als einem Punkt bestehen, lässt sich für Bäume mit abzählbar vielen Enden dadurch erreichen, dass man diesen z. B. Ränder der aus K herausgeschnittenen Gebiete zuordnet. Man erhält so leicht *ebene Kurven, die eindimensional in lauter Kurven oberhalb stetig zerlegt sind.*

Derartig zerlegbare Kurven lassen sich übrigens auch direkt leicht herstellen, z. B. die bei SCHOENFLIES, Bericht II, S. 121.

Die Möglichkeit, eine Kugelfläche oberhalb stetig in Kontinua so zu zerlegen, dass der Zerlegungsraum ein vorgegebener Baum ist, hängt nicht daran, dass eine bloss oberhalb stetige Zerlegung verlangt wird. Sondern

(6) Die Kugelfläche lässt sich (oberhalb und unterhalb) stetig so zerlegen, dass der Zerlegungsraum ein vorgegebener Baum ist.

Diese Möglichkeit beruht darauf, dass eine Kurve auf der Kugelfläche, wie BROUWER [19]) gezeigt hat, gemeinsame Begrenzung von mehr als zwei, ja abzählbar unendlich vielen Gebieten sein kann.

Der Limes einer Punktfolge im Zerlegungsraum ist nun nicht wie bei oberhalb stetigen Zerlegungen der $\overline{lim}$ sup, sondern der $\overline{lim}$ (Näherungsgrenze). Dieser aber ist in kompakten Räumen genau der Limesbegriff des Raums der abgeschlossenen Mengen.[20]) Wir haben also in diesem Raum eine zum gegebenen Baum homöomorphe Menge anzugeben, deren Elemente paarweise fremde Kontinua sind; diese können überdies als Näherungsgrenzen von zu ihnen fremden Kontinuen nur mehr Kurven sein.

Wir verfeinern zur Herstellung der Zerlegung unser ursprüngliches Verfahren auf Grund folgender Tatsachen:

α) Ist B die Begrenzung eines einfach zusammenhängenden Gebietes G auf der Kugelfläche, so gibt es in G zu jeden $\varepsilon > 0$ ein (Kreisbogen-) Polygon P_ε, sodass jeder auf P_ε liegende Punkt um weniger als ε von B und jeder auf B liegende Punkt um weniger als ε von P_ε entfernt ist.[21])

Das bedeutet genau, dass der Mengenabstand $d(P_\varepsilon B) < \varepsilon$ ist.

β) Dieses P_ε kann insbesondere so gewählt, werden, dass die Dicke[22]) des zwischen P_ε und G liegenden Ringgebietes R_ε kleiner als ε ist. Der Mengenabstand zwischen $\overline{R_\varepsilon} = R_\varepsilon + B + P_\varepsilon$ und B ist dann ebenfalls kleiner als ε.

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[19]) Math. Ann. 68 (1910), S. 422—434.

[20]) POMPEJÚ, Ann. Fac. Toulouse (2)7 (1905), S. 281; HAUSDORFF, Grundzüge S. 293 ff.; VIETORIS, Monatsh. f. Math. Phys. 32, 33; TH. WASZEWSKI, Fund. Math. IV, S. 214—245. Als Abstand $d(AB)$ zweier abgeschlossener Mengen A und B verwenden wir die untere Grenze aller $\rho > 0$, für die jeder in A liegende Punkt um weniger als ρ von B und jeder in B liegende Punkt um weniger als ρ von A entfernt ist.

[21]) SCHOENFLIES, Bericht II (1908), S. 114. Vgl. ZORETTI—ROSENTHAL (Enz. IIC 9a), S. 926.

[22]) Dicke von R_ε = obere Grenze der Durchmesser aller in R_ε liegenden Kreisscheiben.

γ_1) Ist G ein von zwei Kurven A, B begrenztes Ringgebiet von einer Dicke $< \varepsilon_1$, so gibt es zu jedem $\varepsilon_2 > 0$ eine endliche Folge von zu einander fremden zweifach zusammenhängenden Kurven (Polygonen) $C_1, C_2, \ldots C_r$, sodass C_i durch C_{i-1} von A, durch C_{i+1} von B getrennt wird, dass $d(A\,C_1) < \varepsilon_2$, $d(C_i\,C_{i-1}) < \varepsilon_2$, $d(C_r\,B) < \varepsilon_2$ ist, dass die Dicke jedes der bestimmten Ringgebiete kleiner als ε_2 ist und dass endlich $d(C_i\,A) < \varepsilon_1$ und $d(C_i\,B) < \varepsilon_1$ ist.

γ_2) Ist G ein Gebiet wie in γ_1), so gibt es zu jedem $\varepsilon_2 > 0$ und jeder Teilung von G durch endlich viele Kurven eine homologe Zerlegung durch Kurven A_j, von denen jede die gemeinsame Begrenzung aller von ihr bestimmten Gebiete ist und von B, bezw. A und B einen Mengenabstand $\leq \varepsilon_1$ hat, sodass ausserdem jedes von einem einzigen dieser A_j in K begrenzte Teilgebiet von G eine Dicke kleiner als ε_2 hat.

δ) Jeder Baum $\mathfrak{R}$ kann für jedes $\varepsilon > 0$ in endlich viele Teilbäume geteilt werden, von denen jeder einen Durchmesser $< \varepsilon$ hat, von denen je zwei höchstens einen Punkt gemein haben *und von denen jeder in $\mathfrak{R}$ höchstens zwei Begrenzungspunkte hat.*

Ist nämlich eine Teilung von $\mathfrak{R}$ gegeben, welche die ersten beiden in δ) genannten Eigenschaften hat, so verbinden wir in $\mathfrak{R}$ je zwei der vorkommenden Begrenzungspunkte durch den einzigen sie verbindenden Bogen. Die Vereinigung dieser Bögen ist ein Teilbaum $\mathfrak{R}_1$ von $\mathfrak{R}$ mit endlich vielen Verzweigungspunkten, alle von endlicher Ordnung. Diese trennen in jedem Element der gegebenen Teilung sämtliche Begrenzungspunkte desselben von einander. Es gibt daher eine Unterteilung der behaupteten Art.

Wir richten nun das Verfahren bei der Konstruktion der stetigen Zerlegung der Kugel K so ein, dass vor Ausführung des n^{ten} Schrittes folgender Zustand erreicht ist.

Der gegebene Baum ist in endlich viele Teilbäume $\mathfrak{R}_1, \mathfrak{R}_2, \ldots \mathfrak{R}_s$ geteilt, von denen jeder einen Durchmesser $< \frac{1}{2^{n-1}}$ hat, von denen je zwei höchstens einen Punkt gemein haben und von denen jeder in $\mathfrak{R}$ höchstens zwei Begrenzungspunkte hat. Die Menge aller dieser möge $\{a_1, a_2, \ldots a_r\}$ sein. Diesem System der a_i, $\mathfrak{R}_l$ ist ein homologes System von Kurven A_i und Teilkontinuen K_l der Kugelfläche zugeordnet, wobei 1) jedes A_i gemeinsame Begrenzung aller von A_i in K bestimmten Gebiete ist, 2) alle K_l eine Dicke $< \frac{1}{2^{n-1}}$ haben, 3) A_i, A_k, sobald sie ein Gebiet K_l gemeinsam begrenzen, einen Mengenabstand $d(A_i A_k) < \frac{1}{2^{n-1}}$ haben.

Der n^{te} Schritt besteht nun in Folgendem: Wir teilen jeden der Teilbäume $\mathfrak{R}_l$ gemäss δ in endlich viele Teilbäume von denen jeder einen Durchmesser $< \frac{1}{2^n}$ hat, von denen je zwei höchstens einen Punkt gemein haben und von denen jeder in $\mathfrak{R}$ höchstens zwei Begrenzungspunkte hat. Dem System der hinzugekommenen Begrenzungspunkte a_j ordnen

wir dann ein bezüglich der schon gewählten Kurven A_l homologes System von Kurven A_j auf K zu, sodass jedes A_j gemeinsame Begrenzung aller von A_j in K bestimmten Gebiete ist, dass die Dicke jedes in K von einem einzigen A_j begrenzten Gebietes kleiner als $\frac{1}{2^n}$ ist und dass der Mengenabstand jedes A_j von der einen, bezw. den zwei Begrenzungskurven von K_l kleiner als $\frac{1}{2^{n-1}}$ ist. (Anwendung von γ_2). Dann wählen wir in jedem von zwei Kurven A_i, A_j oder A_{j_1}, A_{j_2} begrenzten Ringgebiet nach γ_1, so viele zweifach zusammenhängende Kurven (Polygone) A_l, dass sämtliche nunmehr von zwei Kurven A_k begrenzte Gebiete eine Dicke $< \frac{1}{2^n}$ haben und dass je zwei einen solchen Ring begrenzende A_k einen Mengenabstand $< \frac{1}{2^n}$ haben. Und nun wählen wir in $\mathfrak{R}$ als den hinzugekommenen Polygonen A_l entsprechende Punkte a_l irgend welche homolog liegenden Punkte zweiter Ordnung von $\mathfrak{R}$. Damit ist die für den n^{ten} Schritt vorausgesetzte Grundlage für den $(n+1)^{\text{ten}}$ hergestellt. Für den ersten Schritt ist sie vorhanden, wenn wir den Durchmesser der Kugel kleiner als 1 annehmen.

Das einem Punkt y von $\mathfrak{R}$ entsprechende Element Y^* der stetigen Zerlegung von K wird nun genau wie im allgemeinen Fall als Durchschnitt $\prod_{n=1}^{\infty} U_n^y = \overline{lim}_{n=\infty} U_n^y$ definiert, wo $\mathfrak{U}_n^y$ die Vereinigung aller $\mathfrak{R}_l$ des n^{ten} Schrittes ist, in denen y liegt, und U_n^y die zu $\mathfrak{U}_n^y$ homologe Menge in K bedeutet. Ist A_l eine der Begrenzungskurven von U_n^y, so ist der Mengenabstand $d(A_l\ U_n^y) < \frac{1}{2^n} + \frac{1}{2^n} = \frac{1}{2^{n-1}}$. Nun sei A_k eine in U_n^y liegende Begrenzungskurve des m^{ten} Schrittes ($m \geqq n$). Dann hat A_k von mindestens einer Begrenzungskurve $A_{k'}$ des $(m-1)^{\text{ten}}$ Schrittes einen Mengenabstand $< \frac{1}{2^m}$, $A_{k'}$ von einer $A_{k''}$ des $(m-2)^{\text{ten}}$ Schrittes einen Mengenabstand $< \frac{1}{2^{m-1}}$ u.s.w., sodass $d(A_l\ A_k) < \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots < \frac{1}{2^{n-1}}$ und $d(A_k\ U_n^y) < \frac{1}{2^{n-2}}$ ist; d.h. wir hätten Y^* auch als Limes irgend einer Folge $A_{k_1}, A_{k_2}, \ldots$, deren entsprechende Folge $a_{k_1}, a_{k_2}, \ldots$ gegen y konvergiert, definieren können. Nun ist die Abbildung der in $\mathfrak{R}$ überall dichten Menge der Punkte a_l auf die Menge der A_l gleichmässig stetig, weil a_l und a_k für $d(a_l\ a_k) < \frac{1}{2^n}$ in einem $\mathfrak{R}_l$ oder zwei nicht fremden $\mathfrak{R}_l$ des n^{ten} Schrittes liegen und daher $d(A_l\ A_k) < \frac{1}{2^{n-1}}$ ist. Also ist die Menge der Y^*, d.i. $\mathfrak{R}^*$, stetiges Bild von $\mathfrak{R}$, mithin wegen der Eineindeutigkeit der Abbildung und der Kompaktheit von $\mathfrak{R}$ homöomorph zu $\mathfrak{R}$.

Wie im allgemeinen Fall ist $\Sigma \mathfrak{K}^* = K$, womit der Beweis von (6) vollendet ist.

Die Verwendung von δ geschieht zur Erleichterung der Uebersicht und bewirkt vor allem, dass die Verzweigungspunkte von $\mathfrak{K}$ sämtlich als Punkte a_i verwendet werden. Sie ist aber, wenigstens in einfachen Fällen, nicht unumgänglich. So kann man z. B. BROUWERs Konstruktion einer ebenen Kurve, welche gemeinsame Begrenzung aller von ihr bestimmten Gebiete ist, geradezu als die Konstruktion einer stetigen Zerlegung der Ebene in einen Baum $\mathfrak{K}$ mit einem einzigen Verzweigungspunkt auffassen, bei der dieser Verzweigungspunkt nicht als Teilungspunkt im Sinn unserer Konstruktion auftritt, sondern es wird dabei auf jedem Ast $\mathfrak{K}_i$ von $\mathfrak{K}$ eine gegen der Verzweigungspunkt konvergierende Folge von Punkten $a_{i,1}, a_{i,2}, \ldots$ angenommen. Die dazu entsprechenden Kurven der Ebene werden nun so gewählt, dass die Mengenabstände $d(A_{i,n}, A_{j,n})$ mit wachsendem n gegen 0 konvergieren, was durch die immer inniger werdende gegenseitige Umschlingung der $A_{i,n}, A_{j,n}$ erreicht wird.

Ganz wie im Fall der *oberhalb* stetigen Zerlegungen können wir auch aus den *stetigen* Zerlegungen der Kugelfläche *ebene Kurven* herleiten, die (zweidimensional) so stetig zerlegt werden können, dass der Zerlegungsraum eine Kugel ist und *die eindimensional in lauter Kurven (oberhalb und unterhalb) stetig zerlegt sind.* Mit anderen Worten: *Es gibt im Raum der abgeschlossenen Mengen einer Kugelfläche Kurven (nämlich beliebige Bäume) deren Punkte Kurven der Kugelfläche sind und deren Vereinigungsmenge wieder eine Kurve der Kugelfläche ist.* Freilich sind die als Punkte auftretenden Kurven zum Teil eben die merkwürdigen BROUWERschen Gebietsgrenzen, während die Vereinigung als Vereinigung von kontinuierlich vielen paarweise fremden Kurven nach einer Bemerkung von P. URYSOHN [23]) Punkte von der Ordnung $\mathfrak{c}$ haben muss.

Eine eindimensionale stetige Zerlegung der Kugel in lauter im Kleinen zusammenhängende Mengen liefert als Zerlegungsraum einen einfachen Bogen.

Denn infolge der Stetigkeit der Zerlegung kann kein Element derselben einen inneren Punkt enthalten, und weil eine im Kleinen zusammenhängende Kurve gemeinsame Begrenzung von höchstens zwei Gebieten ist, kann der als Zerlegungsraum auftretende Baum nur Punkte erster und zweiter Ordnung enthalten.

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[23]) Mir mündlich von P. ALEXANDROFF mitgeteilt.

Physics. — *"Zur Theorie der Dielektrischen Polarisation in Salzlösungen"*. By LUDWIG EBERT. Supplement N°. 56 to the Communications from the Physical Laboratory at Leiden. (Communicated by Prof. W. H. KEESOM).

(Communicated at the meeting of March 27, 1926).

§ 1. In der letzten Zeit ist von mehreren Autoren versucht worden, die Messungen der D. K. von Ionenlösungen quantitativ theoretisch zu deuten, besonders von BLÜH [1]), HÜCKEL [2]), WALDEN, ULICH und WERNER [3]). Die Veröffentlichung der schönen Versuche der letztgenannten Verfasser war der Anlass zu dem nachfolgend mitgeteilten Erklärungsversuch.

Durch einen Vortrag von Prof. DEBYE in Leyden am 4. 3. 26. wurden mir Ergebnisse einer in seinem Institute ausgeführten Arbeit von H. SACK bekannt [4]); sie behandelt die Wechselwirkung zwischen Ionen und Wasserdipolen durch genaueres Eingehen auf die elektrostatischen Verhältnisse und führt zu einer Gleichung, welche die empirischen Ergebnisse wiederzugeben geeignet ist. Die vorliegende Notiz soll hingegen eine bewusst summarische Darstellung versuchen, nämlich die Zahl derjenigen Wassermoleküle ermitteln, deren vollständiges Verschwinden den beobachteten Effekten äquivalent wäre. Diese Beschreibung soll hauptsächlich zeigen, dass man aus den Messungen der D. K. von Salzlösungen etwa zu denselben Masszahlen einer solchen fingierten "Hydratation" kommen kann, wie man sie auch aus anderen Erscheinungen gewinnt, wenn man auf sie eine ähnlich summarische Absicht anwendet.

§ 2. Der Gedankengang für die Berechnung einer Messung ist folgender. ε_c sei die gemessene D. K. einer Lösung mit c Molen Salz im Liter Lösung; dann ist:

$$\mathfrak{P}_c = \frac{\varepsilon_c - 1}{\varepsilon_c + 2}$$

die spezifische Polarisation der Lösung, eine Grösse, die sich auf 1 ccm bezieht. Hierin ist enthalten:

a. die Refraktion, die von der Anwesenheit des Salzes herrührt. Sie beträgt (pro ccm. Lösung):

$$R \cdot c \cdot 10^{-3},$$

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[1]) O. BLÜH, Ztschr. f. Physik. **25**, 220 (1924).

[2]) E. HÜCKEL, Phys. Ztschr. **26**, 93, insbes. S. 127 ff. (1925).

[3]) P. WALDEN, H. ULICH, O. WERNER, Ztschr. f. Physik. Ch. **116**, 261 (1925).

[4]) Für die Einsichtnahme in diese Arbeit bin ich Herrn Prof. DEBYE und dem Verfasser zu bestem Danke verpflichtet.

wenn R die Molekularrefraktion des Salzes bedeutet, welche entweder bekannt oder abschätzbar ist;

b. die gesamte Polarisation des Lösungswassers. In 1 ccm Lösung ist nicht dieselbe Menge Wasser enthalten, wie in 1 ccm reinen Wassers, sondern die mit dem Faktor f multiplizierte Menge, wenn gilt:

$$f = s^{20}/_{20} - M \cdot c \cdot 10^{-3},$$

wo $s^{20}/_{20}$ die Dichte der Lösung bezogen auf Wasser gleicher Temperatur und M das Molekulargewicht des betreffenden Salzes bedeutet. Wir vergleichen also in ihren Polarisationseigenschaften folgende Grössen, die sich auf gleiche Mengen Wasser beziehen:

$$\mathfrak{P}_c - R \cdot c \cdot 10^{-3} = \mathfrak{P}_c^{H_2O}$$

und:

$$f \cdot \frac{\varepsilon - 1}{\varepsilon + 2} = f \cdot \mathfrak{P}^{H_2O},$$

wo ε und $\mathfrak{P}^{H_2O}$ sich auf reines Wasser beziehen. Die Differenz: $f \cdot \mathfrak{P}^{H_2O} - \mathfrak{P}_c^{H_2O}$ sei $\triangle \mathfrak{P}_c^{H_2O}$ genannt.

Wir machen weiter folgende Annahme, deren summarischer Charakter offenbar ist. Die Differenz: $\triangle \mathfrak{P}_c^{H_2O}$ soll ausschliesslich von dem Verschwinden orientierbarer Dipole herrühren, alle anderen Teile der Polarisation des Wassers [1]) sollen von der Anwesenheit der Ionen unbeeinflusst bleiben. Da man keine experimentellen Erfahrungen über das Verhalten von P_E und P_A unter so eigentümlichen Bedingungen besitzt, wie sie in der nächsten Nachbarschaft eines Ions herrschen können, bleibt nur diese Annahme übrig. Für den wahrscheinlicheren Fall, dass unter diesen Umständen auch $P_E + P_A$ eine Verminderung erfährt, liefert also die eben formulierte Annahme einen *oberen* Grenzwert für die Wirkung der Ionen auf P_0.

Für reines Wasser sei folgende Polarisationsbilanz angenommen:

$$P = P_E + P_A + P_0 = 17{,}3.$$

P_E ist sicher bekannt und gleich 3,7; für den Wert von $P_E + P_A$ hat man zwei Anhaltspunkte:

a. nach RUBENS ist im langwelligsten Ultrarot $n^2 \leqq 4$, woraus folgt: $P_E + P_A = 9$

b. mit der Annahme $P_{Eis} = P_E + P_A$ erhält man den Wert 9,5. Mit dem Mittelwert 9,25 wird:

$$P_0 = 17{,}3 - 9{,}25 = 8{,}05;$$

und der *Orientierungs*anteil $\mathfrak{P}_0^{H_2O}$ in $\mathfrak{P}^{H_2O}$ wird damit für die zu vergleichende Menge Wasser: $f \cdot \mathfrak{P}^{H_2O} \frac{8{,}05}{17{,}8} = f \cdot \mathfrak{P}_0^{H_2O}$.

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[1]) Wegen der hier gewählten Bezeichnungen siehe L. EBERT, Ztschr. f. Phys. Chemie **113**, 1 (1924).

Es ist also $\frac{\Delta \mathfrak{P}_c^{H_2O}}{\mathfrak{P}_0^{H_2O}}$ die relative Verminderung der Orientierungspolarisation in derjenigen Menge Wasser, die sich in 1 ccm Lösung befindet, gegenüber der Orientierungspolarisation derselben Menge Wasser in reinem Zustand bei der gleichen Temperatur. [1])

§ 3. Um aus dieser relativen Verminderung von $\mathfrak{P}_0^{H_2O}$ auf die entsprechende Verminderung der Anzahl orientierbarer Dipole schliessen zu können, sind einige nähere Festsetzungen nötig. Hierbei soll angestrebt werden dem eigentümlichen Charakter des Wassers als *besonders hoch assoziierter Dipolflüssigkeit* Rechnung zu tragen.

Bekanntlich zeigt flüssiges Wasser eine ganz abnorm geringe Orientierungspolarisation $(P_0)_{fl.}$ im Vergleich zu der des Dampfes [2]). Wir nehmen nun zunächst an (*Annahme* A), dies rühre von folgendem speziellen Vorgang her [3]); die Mehrzahl der einzelnen H_2O-Moleküle sind in der Flüssigkeit zu Konfigurationen zusammengetreten, die wegen ihrer höheren elektrischen Symmetrie nicht orientierbar sind (z. B. zu Ringen aus 3 Dipolen, welche aus verschiedenen Gründen nicht ganz unwahrscheinlich sind) [4]). Die gesamte Grösse von $(P_0)_{fl.}$ rühre von dem Rest Monomoleküle H_2O her, welche elektrische Momente gleicher Grösse, wie diejenigen der Dampfmoleküle besitzen sollen. Diese Dipole befinden sich also gleichermassen in verdünnter Lösung innerhalb eines nicht orientierbaren Mediums, ganz entsprechend z. B. den Lösungen von Alkohol- einem Dipolstoff- in Benzol- einem dipolfreien Stoff. Die Konzentration dieser Dipole lässt sich nun folgendermassen abschätzen. $(P_0)_{Dampf}$ ist bei 20° zu etwa 71 anzunehmen; nach DEBIJE ist es für verdünnte Gase und Lösungen bei konstanter Temperatur darstellbar durch:

$$P_0 = c_{Dipol} \cdot \mu^2 \cdot konst., \quad \ldots\ldots\ldots \quad (1)$$

wo μ das Dipolmoment ist. Da $\mu_{Dampf} = \mu_{gelöstes\ Mol.}$ sein soll, ist P_0 ein direktes Mass für die Zahl der in der Flüssigkeit anwesenden Dipole.

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[1]) Da $\frac{\Delta \mathfrak{P}_c^{H_2O}}{\mathfrak{P}_0^{H_2O}}$ in den vorliegenden Versuchen nie grösser wird als $2.5 \cdot 10^{-3}$ und $(1-f)$ noch kleiner ist, braucht man die Vernachlässigung gegenüber $\frac{\Delta \mathfrak{P}_c^{H_2O}}{f \cdot \mathfrak{P}_0^{H_2O} - \Delta \mathfrak{P}_c^{H_2O}}$ nicht zu berücksichtigen.

[2]) Siehe L. EBERT, l.c. die Tabelle auf S. 10.

[3]) Im Grunde müssen die Vorgänge, welche die Verminderung von P_0 verursachen, statistisch behandelt werden. Da die Berechnung der Energiewerte für die einzelnen möglichen Konfigurationen der Wasserdipole noch nicht möglich ist, begnügen wir uns mit den oben festgelegten Annahmen.

[4]) Eine elektrostatische Begründung für die Bevorzugung einer solchen Konfiguration bei kugelförmigen Molekülen siehe z. B. P. DEBYE, Handbuch der Radiologie VI, 636 u. 642 (1924).

Hiernach ist der Bruchteil $\frac{8}{71} = 0{,}113$ der gesamten Zahl von Molekülen H_2O in der Flüssigkeit als orientierbare Dipole H_2O vorhanden, d. h. die molare Konzentration der Dipole in einem Liter reinem Wasser beträgt nach Annahme *A*:

$$c^A_{Dipol} = \frac{1000}{18} \cdot 0{,}113 = 6{,}2.$$

Eine zweite Annahme *B* sei als vielleicht notwendige Ergänzung folgendermassen formuliert. $(P_0)_{fl.}$ soll hier herrühren allein von Doppeldipolen, d.h. Molekülen $(H_2O)_2$, die derart gebaut sind, dass das wirksame Moment $2 \cdot \mu_{Dampf}$ beträgt (Kettenassoziation, siehe L. EBERT, l. c. S. 12). Dass solche oder ähnliche Moleküle mindestens teilweise vorkommen können, kann man aus dem Verhalten des H_2O Dampfes [1]) und aus dem Verhalten der so nah verwandten Alkohole als Dämpfe und in benzolischer Lösung [2]) entnehmen. Offenbar wird nach (1) für diese Doppelmoleküle: $c^B_{Dipol} = 0{,}25 \cdot c^A_{Dipol} = 1{,}55$.

Mit Hilfe dieser Annahmen verfügt man nun über alle notwendigen Grössen, um $\frac{\triangle \mathfrak{P}_0^{H_2O}}{\mathfrak{P}_0^{H_2O}}$ zu deuten; denn da nach (1) $\mathfrak{P}_0$ ein Mass für c_{Dipol} ist, stellt das Verhältnis ein Mass für die relative Verminderung der Dipolzahl in einer bestimmten Menge Wasser dar. Der Ausdruck $c_{Dipol} \cdot \frac{\triangle \mathfrak{P}_c^{H_2O}}{\mathfrak{P}_0^{H_2O}}$ ist dann die absolute Zahl der von *c* Molen Salz festgelegten Dipole.

§ 4. Es folgen nun die Berechnungen für die vier von WALDEN, ULICH und WERNER gemessenen wässrigen Lösungen. Wegen der bekannten Tatsache der grossen Unempfindlichkeit von $\mathfrak{P}$ gegen $\triangle\varepsilon$, sobald ε gross ist, wurden in den nachstehenden Berechnungen auch für die verdünntesten Lösungen alle besprochenen Korrekturen angebracht.

Für reines Wasser wird gebraucht: $\varepsilon = 80{,}50$; hieraus folgt:

$$\mathfrak{P}^{H_2O} = 0{,}96365 \text{ und } \mathfrak{P}_0^{H_2O} = 0{,}444.$$

Die Werte für *R*, *M*, $s^{20}/_{20}$ (bezw. $s^{18}/_{18}$, das als identisch angesehen wird) sind in der Tabelle aufgeführt. Für KCl und $BaCl_2$ sind die Werte von *R* und *s* sicher bekannt, während für die zwei anderen Salze diese Werte wohl mit etwas grösserer Unsicherheit behaftet sind. Die Bezeichnungen in der Tabelle sind oben bereits erläutert ausser der Grösse $w = \frac{\triangle \mathfrak{P}_c^{H_2O}}{0{,}444} \cdot \frac{1}{c}$. Sie ist das Mass für die relative Verminderung der Dipolzahl im Wasser pro Mol Salz. Ist c_{Dipol} die molare Konzentration

[1]) M. IONA, Physik. Ztschr. **20**, 15 (1919); siehe dagegen CH. T. ZAHN, Phys. Rev. **27**, 329 (1926).

[2]) L. LANGE, Ztschr. f. Physik. **33**, 169 (1925).

Uebersicht über die Berechnungen der WALDEN-ULICH-WERNERschen Messungen.

I. KCl. $R = 10.8$; $M = 74.6$; $s^{20}/_{20} = 1 + 486 \cdot c \cdot 10^{-4}$
(LAMB u. LEE, Journ. Am. Chem. Soc. **35**, 168 (1913).

$c \cdot 10^4$	ε_c	$\mathfrak{P}_c$	$(R \cdot c \cdot 10^{-3})10^5$	$\mathfrak{P}_c^{H_2O}$	$(1-f) \cdot 10^5$	$f \cdot \mathfrak{P}^{H_2O}$	$\Delta\mathfrak{P}_c^{H_2O} \cdot 10^4$	$\frac{\Delta\mathfrak{P}_c^{H_2O} \cdot 10^4}{0.444}$	w
1.9	80.20	0.96352	0.21	0.96352	0.5	0.96365	1.3	2.0	1.5
3.7	80.05	342	0.48	342	1	364	2.2	5.0	1.4
5.4	79.60	322	0.58	321	1.5	364	4.3	9.7	1.8
8.0	79.20	304	0.87	303	2.2	363	6.0	13.6	1.7
14.8	78.10	254	1.6	252	4.0	361	10.9	24.5	1.6

II. $BaCl_2$. $R = 21.6$; $M = 208$; $s^{18}/_{18} = 1 + 1840 \cdot c \cdot 10^{-4}$
(A. HEYDWEILLER, nach Land.-Börnst. 5. Aufl. I, 424).

3.4	79.83	0.96334	0.74	0.96333	1.2	0.96364	3.1	7.0	2.1
6.4	79.25	307	1.4	306	2.5	362[5]	5.6[5]	12.8	2.0
12.0	78.27	263	2.6	260	2.9	362	10.2	23.0	1.9

III. $CdBr$. $R = 27.9$; $M = 272$; $s^{18}/_{18} = 1 + 2340 \cdot c \cdot 10^{-4}$ (ibidem).

5.8	80.05	0.96341	1.6	0.96339	2.1	0.96363	2.4	5.4	0.9
11.0	79.88	336	3.1	333	4.1	361	2.8	6.3	0.6
16.2	79.65	325	4.5	320	6.0	359	3.9	8.8	0.5
20.8	79.56	320	5.8	314	8.0	357	4.3	9.7	0.47

IV. $(C_3H_7)_4N \cdot J$. $R = 78$; $M = 313$; $s^{20}/_{20} = 1 + 646 \cdot c \cdot 10^{-4}$
(P. WALDEN, Ztsch. f. phys. Ch. **59**, 400 (1908).

2.0	80.48	0.96363	1.6	0.96361	5.0	0.96360	—	—	—[1]
5.1	79.85	334	4.0	330	12.7	352	2.2	5.0	1.0
7.3	79.65	325	5.7	319	18.7	347	2.8	6.3	0.86
10.9	79.45	316	8.5	307	27.1	338	3.1	7.0	0.64
21.9	78.90	297	16.0	276	54.6[2]	310	3.4	7.7	0.35

[1]) Der erste Wert liefert ein negatives Δ; man darf wohl vermuten, dass es sich um einen Messungsfehler in ε_c handelt.

[2]) Der Vergleich dieser Spalte mit den anderen Salzen ist lehrreich; man sieht, dass hier in 1 ccm Lösung bereits 0.5 % weniger Wasser ist als in 1 ccm Wasser. Die grossen Ionen nehmen ein merkliches Volumen ein, ohne durch starke Kontraktion die Differenz soweit ausgleichen zu können wie oben, wo $1-f$ den Betrag von 1 ‰ nicht erreicht.

der Dipole in reinem Wasser, so ist also $w \cdot c_{Dipol}$ die von einem Mol Salz fixierte Zahl Mole der vorhandenen Dipole.

§ 5. Die Besprechung der Ergebnisse betrifft einerseits die absolute Grösse der Zahl fixierter H_2O-Moleküle, andererseits das Verhältnis der Grössen w untereinander, sowie die teilweise vorhandene Abhängigkeit der Zahlen w von c.

Die *absolute Zahl* der fixierten Dipole pro Mol Satz ist $w \cdot c_{Dipol.}$ d.h. wird etwa zwischen $w \cdot 6{,}2$ und $w \cdot 1{,}55$ liegen, die Zahl der H_2O-Moleküle (s. o. § 3) also zwischen $w \cdot 6{,}2$ und $w \cdot 3{,}1$. Sie übersteigt demnach in allen betrachteten Fällen die Zahl 12 nicht wesentlich.

Man wird hierdurch in den Stand gesetzt, die elektrostatische Wechselwirkung zwischen Ionen und Wasserdipolen mit einem wirklichen Wegnehmen von Molekülen aus dem Lösungsmittel zu vergleichen; denn alle vorliegenden Versuche, die aus anderem Material einen — wenigstens angenäherten — Schluss auf diese Menge zulassen, führen auf etwa ebenso grosse Zahlen. In diesem Sinne kann man z. B. die Aussalzwirkungen auf dipolfreie Gase — z. B. C_2H_2 — so deuten, als ob das Salz eine bestimmte Menge von H_2O-Molekülen aus der verfügbaren Menge Lösungsmittel wegnähme. Aus ihren sorgfältigen und ausgedehnten Messungen gewinnen so z. B. MANCHOT und seine Mitarbeiter [1]) Zahlen, die für KCl etwa 9 betragen u. s. w. Es erscheint lohnend, sobald umfassenderes Material über die *DK* von Salzlösungen vorhanden ist, diese Erscheinungen genauer mit w zu vergleichen. Ausserdem muss man auf die qualitativ als sicher anzusehende Tatsache hinweisen, dass bei der Bewegung der Ionen Wasser mitgeführt wird. Einen gewissen Anhaltspunkt für die Menge dieses Wassers kann man hier durch Anwendung des STOKES'schen Gesetzes erhalten. Da dieses bei grossen organischen Ionen gewöhnlich zu bemerkenswert vernünftigen Werten für die Grösse der bewegten Teilchen führt, darf man wohl auch erwarten, dass es für die einfachen Ionen, bezw. ihre Aggregate mit Wassermolekülen (mit Ausnahme der Sonderfälle des H^+ und OH^-Ions) kein ganz absurdes Ergebnis liefert [2]). Die hiernach berechneten scheinbaren Radien sind stets von der Ordnung weniger Å.E., können also sehr wohl als einigen mitgeführten H_2O-Molekülen äquivalent angesehen werden. Diese Bemerkung gilt übrigens für alle Grössen und Parameter, die in der sehr ausgedehnten einschlägigen Literatur als Radien gelöster Ionen auftreten. — Aus Ueberführungsmessungen kann man im Allgemeinen nur auf die relativen Unterschiede der Wassermitführung mehrerer Ionen schliessen; doch sind neuerdings auch Versuche einer absoluten Messung bekannt

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[1]) W. MANCHOT, M. JAHRSTORFER und H. ZEPTER, Ztschr. f. anorg. u. allg. Ch. **141**, 45 (1924).

[2]) R. LORENZ, Ztschr. f. Physik. Ch. **73**, 252 (1910), und das Buch: „Raumerfüllung und Ionenbeweglichkeit"; P. WALDEN, Ztschr. f. anorg. u. allg. Ch. **113**, 125 (1920).

geworden, die für das besonders stark sich betätigende Ion Li^+ nur die Mitführung von etwas mehr als 10 Molekülen Wasser ergeben. —

Es scheint also möglich zu sein mehrere recht verschiedene Erscheinungen der Grössenordnung nach durch einen einzigen einfachen Vorgang in allerdings roher Weise verständlich zu machen, ohne Einführung besonderer neuer Annahmen. So nötig es ist eine bestimmte Einwirkung der Ionen auf das Lösungsmittel anzunehmen, so muss man doch andererseits gebührend berücksichtigen, wie wenig die physikalischen Eigenschaften der verdünnten Salzlösungen sich von denen des reinen Wassers unterscheiden. Erinnert sei z.B. an die Zähigkeit von Lösungen einfacher Salze, die in 0,01 mol. Lösung für KCl nur etwa um 0,1% von der des Wassers sich unterscheidet. Das Wasser ist ja kein besonders günstiges Medium für Äusserungen polarer Einwirkungen; hierfür spricht schon die hohe D. K. und ausserdem sind die Dipole hier abnorm stark assoziiert und die entsprechenden Wirkungen, auf Mole bezogen, werden in „normaleren" Dipolflüssigkeiten, wo auch ε kleiner ist, zweifellos viel stärker zum Ausdruck kommen [1]). Da P in den hier wichtigen Fällen äusserst unempfindlich ist gegen $\triangle\varepsilon$, muss man allerdings zur genügend sicheren Abschätzung der Zahlen w über gute Messungen von R und s in den betreffenden Fällen verfügen, was heute im Allgemeinen noch nicht der Fall ist. Ausserdem kommen bei Lösungsmitteln mit kleinem ε Assoziationen der Ionen in Frage, die eine ausserordentlich starke Erhöhung von P_0 bedingen können.

§ 6. Wir wollen noch die *relative Grösse* der Zahlen w besprechen, obwohl das bisher veröffentliche Material für eine solche Diskussion durchaus noch nicht die wünschenswerte Ausdehnung besitzt.

Als einfachste Fälle zeichnen sich KCl und $BaCl_2$ aus; denn hier ist w innerhalb der Messungsfehler konstant. $w_{KCl} : w_{BaCl_2}$ ist etwa gleich 1,6 : 2,0. Dieses Verhältnis scheint gut zu unseren Vorstellungen über die Grösse der nackten Ionen zu passen, wenn man annimmt, dass die Wirksamkeit des Cl Ions etwa die Hälfte der des K-Ions beträgt. Wenn man als Mass der Wirkung eines Ions den Ausdruck $\frac{l}{r^2}$ einführt und die von GRIMM gegebenen Zahlen für das Verhältnis $\frac{r_{Ba^{++}}}{r_{K^+}}$ benutzt, [2]) so sieht man, das die Wirksamkeiten des K^+ und Ba^{++}-ions etwa gleich sein sollten. Mit den Masszahlen $w_{Cl^-} = 0{,}5$ und $w_{K^+} = w_{Ba^{++}} = 1{,}0$ erhält man etwa die oben abgeleiteten Zahlen für diese Salze.

Dass in diesen Verdünnungen die Wirkung konstant ist, ist nur zu

[1]) Hierin sehe ich die Hauptgründe für die abnorm grossen scheinbaren Radien der Atomionen in nichtwässrigen Lösungen (P. WALDEN l.c.) und für den offenbar viel grösseren Einfluss gelöster Ionen auf die Zähigkeit (CH. A. KRAUS, Properties of electric conducting systems, New-York, 1922, S. 112).

[2]) H. G. GRIMM, Ztschr. f. Physik. Ch. 98, 372 (1921).

erwarten, da auch die thermodynamische Aktivität des Wassers in diesen Lösungen praktisch von c unabhängig ist.

Diese Überlegung ist auch massgebend für die Auffassung der Veränderlichkeit von w mit c in den beiden anderen Fällen; sie muss auf Vorgänge zwischen den gelösten Teilchen zurückgeführt werden. Bei $CdBr_2$ ist eine solche Erklärung ausserordentlich naheliegend, da Cd-Salze durch Komplexbildung ausgezeichnet sind. Diese bewirkt stets eine Verminderung der gesamten Ionenzahl und das zunächst möglicherweise entstehende Ion $[CdBr]^+$ ist grösser und ausserdem weniger hoch geladen als Cd^{++} und der dann folgende Komplex $[CdBr_2]$ ist neutral, wodurch sich die zunehmende Verminderung von w verstehen lässt. Nicht so leicht bietet sich eine Erklärung für das letzte Beispiel. Dass hier w sehr klein ist, leuchtet wegen der Grösse der beiden Ionen ein; die Abnahme von w kann — sofern alle experimentellen Unterlagen als durchaus sicher angenommen werden — wohl nur auf eine gegenseitige Assoziation der Ionen als solcher hindeuten; chemische Vorgänge werden wegen der Stärke der den Salzionen entsprechenden Säure und Base nicht in Frage kommen. Diese Tatsache macht die leider noch ausstehende genaue Untersuchung der thermodynamischen Eigenschaften dieser Salzlösung besonders wünschenswert. Dass ein sehr grosses organisches Ion in der Abhängigkeit seiner Aktivität von c sich merklich von den einfacheren Ionen unterscheidet, ist allerdings schon früher gelegentlich festgestellt worden [1].

Leiden, 24. 3. 1926.

[1] H. v. HALBAN u. L. EBERT, Ztschr. f. Phys. Ch. **112**, 384 (1924).

Mathematics. — *"The normal, not absolutely closed topological space".* By WILFRID WILSON. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of March 27, 1926).

1. The main object of the present paper is to prove the theorem:

Any normal, not absolutely closed topological space $\mathfrak{R}$ can be extended to a normal topological space $R = \mathfrak{R} + \xi$ by adjunction of a non isolated point ξ.

I am informed by Prof. ALEXANDROFF that a proof of this theorem resting on entirely different foundations has also just been given but not yet published by Mr. TYCHONOFF. The following investigation was suggested by the concluding remarks in the work of ALEXANDROFF and URYSOHN. — "Zur Theorie der topologischen Räume" [1]).

2. By a topological space is understood any set satisfying the axioms (A), (B), (C) and (D) of HAUSDORFF [2]).

A point x is said to be regular [3]) in the space $\mathfrak{R}$ if for each neighbourhood $U(x)$ a neighbourhood $V(x)$ exists such that $\overline{V}(x) \subset U(x)$, where $\overline{V}$ represents V together with all boundary points of V.

The space $\mathfrak{R}$ is said to be regular when it contains only regular points.

A space $\mathfrak{R}$ is said to be normal [4]) when to any two closed sets F_1 and F_2 without common points, regions (open sets) U and V exist such that $U \supset F_1$, $V \supset F_2$ and $UV = 0$.

A topological space $\mathfrak{R}$ is said to be absolutely [5]) closed when it is impossible to adjoin a point ξ in such a way that $\mathfrak{R} + \xi$ is a topological space in which ξ is not isolated containing $\mathfrak{R}$ and regions in $\mathfrak{R}$ as regions in $\mathfrak{R} + \xi$.

3. Let $\mathfrak{R}$ be a normal, not absolutely closed topological space. Then there exists a system $V = \{G\}$ of regions covering $\mathfrak{R}$ and such that any finite subset $G_1, G_2, \ldots, G_n$ of V satisfies the condition

$$\mathfrak{R} - (\overline{G_1} + \overline{G_2} + \ldots + \overline{G_n}) \neq 0. \text{ [6])}$$

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[1]) Math. Annalen, Bd. 92, 3/4 Heft.

[2]) "Grundzüge der Mengenlehre", Leipzig, 1914, S. 213.

[3]) VIETORIS, "Stetige Mengen", Monatshefte f. Math. u. Phys. 1921, S. 173. ALEXANDROFF und URYSOHN, Topologische Räume, Math. Ann. 92, S. 263.

[4]) ALEXANDROFF und URYSOHN, [3]), S. 263.

[5]) ALEXANDROFF und URYSOHN, [3]), S. 261.

[6]) ALEXANDROFF und URYSOHN, [3]), S, 262, Satz II.

Consider all pairs (x, G_π) where x is an arbitrary point of $\mathfrak{R}$ and G_π an arbitrary region of the system V containing x. $\mathfrak{R}$ being normal we can assign to each pair (x, G_π) a system G_π^τ, $(0 \leqslant \tau \leqslant 1)$, of neighbourhoods such that

$$G_\pi^1 = G_\pi \text{ and } x \subset G_\pi^{\tau_1} \subset \overline{G_\pi^{\tau_1}} \subset G_\pi^{\tau_2}, [7]) \ (\tau_1 < \tau_2).$$

To the space $\mathfrak{R}$ adjoin the point ξ and to ξ assign the neighbourhoods

$$U_{\substack{\tau_1, \tau_2, \ldots, \tau_n \\ \pi_1, \pi_2, \ldots, \pi_n}}(\xi) = \xi + \mathfrak{R} - (\overline{G_{\pi_1}^{\tau_1}} + \overline{G_{\pi_2}^{\tau_2}} + \ldots + \overline{G_{\pi_n}^{\tau_n}})$$

where the G_{π_i} are components of arbitrary pairs $(x_i, G_{\pi_i}), \tau_i \neq 1, (i = 1, 2, \ldots, n)$.

4. $\mathfrak{R} + \xi = R$ is a topological space.

(A). Each point x of R has a neighbourhood containing x.

(B). The intersection of any two neighbourhoods U and V in R is a neighbourhood, for

a. If U and V are in $\mathfrak{R}$, UV is a neighbourhood since $\mathfrak{R}$ satisfies axiom (B).

b. Let $U(\xi) = U_{\substack{\tau_1, \tau_2, \ldots, \tau_n \\ \pi_1, \pi_2, \ldots, \pi_n}}(\xi)$ and $U_*(\xi) = U_{\substack{\tau_1^*, \tau_2^*, \ldots, \tau_m^* \\ \pi_1^*, \pi_2^*, \ldots, \pi_m^*}}(\xi)$ be any neighbourhoods of ξ. Then

$$U(\xi)\, U_*(\xi) = U_{\substack{\tau_1, \tau_2, \ldots, \tau_n, \tau_1^*, \tau_2^*, \ldots, \tau_m^* \\ \pi_1, \pi_2, \ldots, \pi_n, \pi_1^*, \pi_2^*, \ldots, \pi_m^*}}(\xi)$$

(C). It follows from the definition of the $U(\xi)$ that $U(\xi)\ \mathfrak{R}$ is a neighbourhood in $\mathfrak{R}$ of any of its points.

(D). Let $x \neq y$.

a. If $x + y \subset \mathfrak{R}$ axiom (D) is satisfied since $\mathfrak{R}$ is a topological space.

b. If $y = \xi$ let G_π be that neighbourhood of the system $\{G\}$ containing x. Then (x, G_π) is one of the pairs (x, G_π). Choose $U(x)$ so that

$$U(x) \subset G_\pi^\tau \subset \overline{G_\pi^\tau} \subset G_\pi \text{ and } U(\xi) = U_\pi(\xi).$$

Then $U(\xi)\, U(x) = 0$.

Finally ξ is not isolated in R since

$$\mathfrak{R} - (\overline{G_{\pi_1}^{\tau_1}} + \overline{G_{\pi_2}^{\tau_2}} + \ldots + \overline{G_{\pi_n}^{\tau_n}}) \supset \mathfrak{R} - (\overline{G_{\pi_1}} + \overline{G_{\pi_2}} + \ldots + \overline{G_{\pi_n}}) \neq 0$$

since $G_{\pi_1}, G_{\pi_2}, \ldots, G_{\pi_n}$ belong to the system V. Hence $U_{\substack{\tau_1, \tau_2, \ldots, \tau_n \\ \pi_1, \pi_2, \ldots, \pi_n}}(\xi)$ contain points other than ξ.

5. R is regular in the point ξ, for to any

$$U_{\substack{\mu_1, \mu_2, \ldots, \mu_n \\ \pi_1, \pi_2, \ldots, \pi_n}}(\xi) \text{ we have a } \overline{U_{\substack{\tau_1, \tau_2, \ldots, \tau_n \\ \pi_1, \pi_2, \ldots, \pi_n}}(\xi)}$$

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7) URYSOHN, "Zum Metrisationsproblem", Hilfssatz, Math, Ann. Bd. 94, S. 310.

so that $\overline{U_{\substack{\tau_1,\ldots,\tau_n \\ \pi_1,\ldots,\pi_n}}(\xi)}$

$$\subset \xi + \Re - (G_{\pi_1}^{\tau_1} + G_{\pi_2}^{\tau_2} + \ldots + G_{\pi_n}^{\tau_n})$$

$$\subset \xi + \Re - (\overline{G_{\pi_1}^{\mu_1}} + \overline{G_{\pi_2}^{\mu_2}} + \ldots + \overline{G_{\pi_n}^{\mu_n}})$$

$$= U_{\substack{\mu_1,\mu_2,\ldots,\mu_n \\ \pi_1,\pi_2,\ldots,\pi_n}}(\xi), \quad (\tau_i > \mu_i).$$

6. R is normal [8]).

Let the closed sets F_1 and F_2, each containing more than one point, be such that $\xi \subset F_1$ and $F_1 F_2 = 0$. Let $U_1(\xi)$ be a neighbourhood for which $F_2 \overline{U}_1 = 0$, define $V_1 \equiv R - \overline{U}_1$ and $F_3 \equiv F_1 (R - U_1)$, so that $V_1 \supset F_2$ and $U_1 V_1 = 0$. Because $F_2 \subset \Re$, $F_3 \equiv F_1 (R - U_1(\xi)) \subset \Re$, and $\Re$ is normal, neighbourhoods $U_2 \supset F_3$ and $V_2 \supset F_2$ exist for which $U_2 V_2 = 0$. Then

$$(U_1 + U_2)\, V_1 V_2 \subset U_1 V_1 + U_2 V_2 = 0, \text{ and } U_1 + U_2 \supset F_1, \; V_1 V_2 \supset F_2$$

from which the normality of R follows. In the last paragraph we have proved the theorem:

The adjunction of a regular point to a normal space gives a normal space.

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[8]) I am indebted to Prof. ALEXANDROFF for the suggestion that R is normal.

Anatomy. — *"On the Problem of Anthropogenesis."* By Prof. L. BOLK.

(Communicated at the meeting of December 19, 1925).

Since the conviction began to make its way, that our form has gradually developed from another more primitive one, the study of this form has ever been of a more or less biassed nature. The investigation into specifically human characteristics for their own sake, or an inquiry into the essentially morphological character of the human physical properties, was made subordinate to a study of these characteristics with a preconceived purpose. Considered as elements of comparative anatomy their value was tested for the construction of a pedigree for Man. The measure, by which that value was determined, was the greater or lesser correspondence to the degree of development of the characteristic under consideration in forms, that had been assumed beforehand as being more primitive. Comparative anatomy was studied with a certain tendency: genealogy and the problem of the descent predominated over the study of the form. Far be it from me indeed to deny the immense profit science has derived from the views obtained along this path.

But the one-sided application of this method implies a danger, viz. that the study of the form as a phenomenon and of its specific qualities in their interrelations, is somewhat neglected. But, you ask me, can it be worth while trying to understand the form of the human body apart from any hypothesis whatever of the descent of man? For there is no denying the fact that also the form of our body has been developed from a series of constantly changing forms, our primitive forms, and that, to confine ourselves to the Primates the series of forms in the phylogenic of the now living Primates must at some point in the part meet the line of descendance of Man. I do not deny this at all, but two sides of the question should be kept distinct, viz. that of the structure of the primitive human form — this is the anatomical side of the problem; and secondly, that of the etiological factor by which Man obtained his present shape — this is the physiological side of the problem. It seems to me that one should begin by answering the second question, and must try to find an answer by studying the human form itself.

At variance with the methods employed, I purpose to discuss in the present paper the historical genesis of our actual form, entirely apart from any theory of descent. I was prompted to do so by the ever-growing conviction that the solution of this problem cannot be arrived at by applying the current methods. Just as it would be impossible for me to obtain a true insight into the degree of relationship, which undeniably exists between myself and every one here present, by comparing only our individual bodily characteristics, so would it be impossible to conclude

anything about the relationship between present-day Man and the other Primates now living by comparing our specific characteristics with those of the anthropoids or lower monkeys in existence at present. And the greater or lesser resemblance in form is no reliable index, because it is not of necessity a consequence of relationship, but may be effected by the same etiological factor, that has operated repeatedly. To exemplify this in a simple way, I will assume the possibility that in some of you, for instance, the lateral incisor of the upper jaw be lacking ; this does not imply a closer relationship between the individuals concerned, but is the result of a cause having acted independently upon their individual development. Now that, which holds for individuals of one species, is also applicable to species belonging to a genus, and to genera belonging to the same family. Nobody indeed would consider the absence of a tail, or a thumb, in Primates as an indication of relationship ; well then, why should we attach such great importance to the presence or the absence of a cusp on the crown of a molar ? If Man possessed a dentition in which one might almost vainly look for variations, as in the case of some species of Primates, such a characteristic might be a trustworthy guide. But whoever has had the occasion to view the extraordinary variety in the cusps of human molars must be convinced that in this case the teeth are a most unreliable guide for any conclusions as to the degree of relationship. What criterion have we, moreover, upon which to base an assumption of relationship upon homogeneity of form? Absolutely none!

Furthermore, as it is impossible to deduce the individual properties of my own skeleton from those of the skeleton of one or more of my immediate ancestors, so is it impossible to deduce the specific properties of the human skeleton in general from skeletal remains of extinct types found by mere chance. For, just as every individual is a new creation, a creature sui generis, a result of heredity and individual genesis, so is every species a new creation in which what is inherited, is modified through the evolutional factor, which I believe to be essential to life itself. Evolution is not a result but a principle; it is a function of life, it is for the organized world as a whole what growth is for the individual, but, just as in the individual development, it is subject to external influences.

Comparative anatomy can catalogue the structural differences between now living Primates and Man and the structural variations in paleontological objects ; they can be arranged systematically like the words in a dictionary, and each new discovery means a lengthening of the list. But the richer the store of words becomes, the greater will be the possibility of constructing sentences with the aid of these words. Each new discovery increases the number of possibilities. And, although it may sound paradoxical, it is my firm conviction that the more numerous the discoveries of extinct forms, the more shall we be convinced of the impossibility of solving the problem of the genesis of the human form in this way. This does not in the least detract from the great significance of the

paleontological material in general for tracing the principal line in the historical development of certain groups, especially when these are characterized by a pronounced specificity. I merely wish to point emphatically to the inadequacy of this method for application to a form so little differentiated as that of Man.

Let us, therefore, abandon the deductive method used to lift the veil from the genesis of our form, and let us see if the application of an inductive method can serve our purpose better. To this end our form should not be the goal but the starting-point of our inquiry. Beginning with Man himself, we should, independent of any consideration of relationship, commence by answering the question: What is the essential of Man as organism, and what is the essential of Man as form? As you will see the character of my question is twofold, physiological and anatomical. We have to consider Man as a being and Man as a phenomenon. This does not imply a real contradistinction, but a methodological principle. For our bodily form is the result of our development, and this is a function of the organism. It will appear from my further argumentation that the same causality forms the basis to the essential of man as organism and as form.

Before proceeding to answer the above questions, let it be pointed out that the problem of Anthropogenesis, as I conceive it, is more comprehensive than it has hitherto been conceived to be, on account of the physiological element implied in it. Hitherto the anatomical side of the question was always put in the foreground, the appreciation of the somatic differences between adult forms of the Primates was the clue to a conception. I, on the contrary, will try to approach the solution of the problem along the physiological way, tracing the essentials of our form from the specific characteristics of our genesis and the character of this genesis as an expression of the specifically human as organism.

To trace how my present conception of the Anthropogenesis has developed in the course of time, I must go back to a short communication made more than 25 years ago, dealing with the observation of several anomalies in an adult man, which had all something in common. I found in this individual a number of anatomical peculiarities which occur transitorily in the normal development of the human fetus, but which had persisted in this individual. I described this case as a curiosum, without attaching to it a significance of a more general nature. It was merely a description. But now I wish to point to something that is psychologically remarkable; in my sub-consciousness the idea: „persistence of fetal characteristics" must have been implanted by this observation, and involuntarily it must have influenced my conception of specifically somatic features of Man. For in later researches the idea crops up repeatedly in one form or other. Thus, for instance, in a conclusion after an inquiry into the phylogenetic development of the human dentition, in which I demonstrated that our first permanent molar is originally a milk

tooth that has become permanent and is thus an infantile, i.e. temporary, organ which has changed into a permanent organ. Afterwards this idea took on a more distinct form in a comparative investigation into the position of the Foramen magnum in Man and in monkeys. Quite in accordance with the usual scheme, in which the monkey-skull is considered as the primitive one from which the human skull is derived, the current view was that the Foramen magnum in Man has shifted anteriorly in conformity to the erect posture. I came to the contrary conclusion: that the central position of the Foramen magnum in Man is the primitive one with all Primate-fetus, but that in the other Primates it shifts to the posterior pole of the skull, whereas in Man it persists in this position. Here the idea: persistence of a fetal condition was applied for the first time not as an individual variation but as etiological factor of a normal, specifically human feature. What was sub-conscious before, had now become an element of a conscious association. However it was only an isolated case as yet, there was no question of generalization of this causality. But no idea brands itself more strongly upon the human mind than the one that has been conceived as an objection to an existing conception. What in our mental mechanism was first a conclusion, becomes a touchstone for the possibility as to whether other phenomena may not be accounted for in the same way. Should this appear to be the case, the supposition arises whether perhaps all properties can be explained from this new point of view; further investigations raise this supposition to probability, criticism leads to certainty, and along this path a causal principle develops in our mind from a conclusion.

And in this way, in the course of years, the idea: persistence of fetal characteristics, from being a simple characterization of an observed condition has become to me the basis for a comprehension of the human somatic form. For our essential somatic properties, i.e. those which distinguish the human body from that of the other Primates, have all one feature in common, viz. they are fetal conditions that have become permanent. What is a transitional stage in the ontogenesis of other Primates has become a terminal stage in Man. His smooth skin, the form of his skull, the central position of the Foramen magnum, the shape of his ear, his unpigmented skin, the presence of labia majora, the structure of hand and foot, etc., all these can be also observed as temporary conditions of the fetus of every Primate, but get lost in the further individual development, being succeeded by other conditions. This is why the fetus of monkeys has in general a much more human aspect, not, as would follow from an application of the biogenetic law, because monkeys are descended from more man-like ancestors, but because man has preserved the fetal type up to his adult state. The morphogenesis of the other Primates goes further than that of Man. This difference between men and monkeys I will indicate by characterizing the development of Man as conservative and that of the other Primates as propulsive.

The recognition of this fact could not fail to alter entirely my conception of the relation between the human somatic properties and those of the other Primates, but at the same time, and that means much more, it was sure to bring about a change in my views regarding the causality of Anthropogenesis.

For, as regards the first point, I was now able to answer the fundamental question propounded at the outset of this paper: What, then, is the essential in the anatomical characteristic of the human form? The answer is: The fetal character of his form. And this opens up a point of view upon that form and its genesis quite apart from any consideration of relationship or pedigree-construction, as it appears that we need not of necessity deduce the specific properties of our body from those of other Primates. For where present-day Man may be bodily considered as a Primate-fetus that has become sexually mature, our ancestors already possessed all our specific properties, though only as temporary developments, by way of transitional stage. This is the principle of what I term the *Fetalization-theory* of Anthropogeny.

As stated before, this understanding of our form brought about an essential change in my conception of its genesis, and that in two respects.

When all the characteristics of Man have something in common, it is not conceivable that they should have originated independently of each other, each from a cause of its own. Like effects must result from a common cause. The body as a whole was transformed because its development stopped at a younger stage, a process that was brought about gradually of course. The cause must therefore have operated uninterruptedly for a period of time that cannot be determined approximately. But from this it follows again that this fetalization cannot have resulted from external influences acting upon the organism. It was not the effect of an adaptation to modified external conditions, it was not brought about under the influence of a struggle for life, it was not the result of natural, or sexual selection, for these evolutional factors exert their influence not on the body as a whole but in a more limited way, on circumscribed parts of the organism. Thus the etiological factor of the process must have been an internal, a functional one. In short, an anthropogenesis as consequence of a single, organic principle of development. Anyone conversant with the tenor of the current theories on this point will readily understand the difference between the point of view I have gradually come to adopt and the principle of these theories. An elaboration of my theory would have to commence by dividing the human properties into two groups, viz. primary and consecutive properties. As an instance of the former, I might mention Man's glabrate skin; of the latter, the strong differentiation of the muscles of his face.

The human form, thus, as the effect of a single, functional cause. What can have been the nature of this cause?

When fetal qualities or conditions become permanent, this must be due

to an influence that prevents these qualities from reaching the proper degree of development; they remain in a transitional stadium, more or less removed from their original terminal stage. Consequently a restraining influence must come into play. Thus we can look upon the human form as the effect of a general restraining of development.

When this had once become clear to me, the question naturally arose as to whether this functional principle applied only to the development of the human form, because the morphogenesis, as I have already pointed out, is only one of the functions of the organism. What about the other functions?

On further consideration of this problem it gradually became clear to me that the restraining principle put its stamp not only on the morphogenesis, i.e. not on the developmental progress alone, but on the entire individual life-progress of Man. When we compare Man in this respect with the other Mammals, and particularly with the Primates, we perceive at once that there is no Mammal that grows so slowly as Man, and not one in which the full development is attained at such a long interval after birth. And, though on this point no time-data are at our disposal, it is nevertheless sure, that the organisms nearest to Man, the Anthropoids, are also far behind him as regards the duration of their prime period. This slow development, this protracted maturity, is eventually succeeded by a phase of decay which again proceeds more tardily than in the case of any other Mammal. I cannot go into a detailed argumentation and must confine myself to general remarks, as I must get to the point I am aiming at and my final conclusions as quickly as possible along the main lines of my theory. But what has just been said furnishes us with a startingpoint from which to answer the second question: What is the essential in Man as an organism? The obvious answer is: The slow progress of his life's course. Again I cannot produce special evidence for my assertion that this slow tempo is the result of a retardation that has gradually come about in the course of ages. This I will term the *Retardation-hypothesis* of Anthropogenesis. This retardation is demonstrable on historical grounds. Rather more than a year ago I demonstrated in a communication read before this assembly that, for instance, prehistoric Man, Homo neanderthalensis, developed at a more rapid rate than Home recens. Another proof of this retardation is the fact that not all living races of the genus Homo have an equally retarded course of life. Furthermore this retardation accounts for a number of remarkable facts, and in part also for contradictions in the functional development of the human organism. But of all this I must not speak, though to my extreme regret, for you can well imagine how keen a delight I should take in reviewing Man in his form and in his nature in the light of these principles.

So it is evident that, with respect to his form and his organism, Man is distinguishable by a very particular characteristic, viz. the essential in his form is the result of a fetalization, that of his life's course is the result of

retardation. These two facts are closely related, for, after all the fetalization is the necessary consequence of the retardation of the morphogenesis. Anatomically this effect is brought about in two ways, namely, it may be that a fetal characteristic persists, because, as is demonstrable in other Primate-fetus, its disappearance is delayed to a later phase of development, until at last it remains constant. As an example of this I may refer to the labia majora of the woman. Also in the lower Primates we find rudiments of these pudenda, but very early they disappear here; in Anthropoids they also disappear, but not entirely until after birth; in Man finally they persist. Here, then, we see the disappearance of a fetal characteristic, first retarded, then lacking. The second way in which, anatomically, the form is fetalized through retardation is of more frequent occurrence. As an example I shall take the hirsuteness. In the fetus of the lower Primates the hairy coat is present at a comparatively early stage. In that of the gibbon the development is already retarded, the neo-natus has hair only on skull and dorsum, the belly is bare or but slightly covered with hair, the hairy coat appearing after birth. With anthropoids there is hair only on the skull of the fetus, just as is the case with Man, for the rest they are glabrate, and acquire the hairy coat only in the course of the first half-year of life. The breast of Gorilla remains glabrate throughout life. In Man, finally, the skull only is covered with hair, and for the rest the fetal skin remains glabrate. Here, then, the retardation has reached its maximal intensity, i.e. a standstill, and has ended in elimination of a characteristic.

I need hardly emphasize that the above exposition of the process of Anthropogeny is still incomplete. I have set forth how Man originated through a modification — a retardation — of the common life's course which entailed a fetalisation of the form of his body. But we have now to consider the final question: What was its immediate preceding cause. I purposely say its immediate preceding cause, for the possibility of this cause, whatever it is, being spontaneous is a priori precluded. This cause also must have been the effect of a preceding one, and the anthropogenesis itself must ultimately be considered as a symptom of the evolutional principle of life. Indeed, I have already suggested that, in my opinion, evolution is an inherent function of life as such, of organism itself; to inquire into its cause is equivalent to seeking for the nature of life itself.

My answer to the question I have propounded shall be brief: The gradual retardation of the life's course of the ancestors of Man with all the consequent effects, both as regards his morphological features and his functional properties, must have had for its immediate cause a modification of the action of the endocrin system of the organism. This system built up of a number of organs distributed over the body, regulates the metabolism.

During the first phase of our life it governs the morphogeny, later the

maintenance of the form. Again I must pass over in silence how our body with its multiplicity of properties develops through the action of the hormones, secreted by these organs. This would furnish material for a separate discussion. Suffice to say that in controlling the intensity of the metabolism these hormones can inhibit or promote the growth. In Anthropogenesis an inhibitive action appears to have gradually increased in significance, the rate of development became slower, the progress of development more retarded, and this influence revealed itself as much with regard to the development of the organism in toto as with regard to the genesis of separate morphological properties.

Though all this brings out the great significance of the endocrin system for the genesis of our form, it does not by any means open up a new biological vista. For already numerous experiments have shown us that in lower forms a retardation or acceleration of development can be effected by the addition of hormones to the food. In this connection I may remind you of the discourse held by our fellow-member VERSLUYS a few months ago at a meeting of our Academy. But now I utilize this fact, as biological basis for the historical, normal development of a form so eminently differentiated as that of Man from his primitive parents, and use it as a starting point for the solution of the highly important problem of Anthropogenesis. In passing I wish to point out that fetalization is, therefore, related to the phenomenon known in biology as Neoteny.

Time is lacking for me to bring forward any special argumentation. Moreover, this would lead me into the field of embryology and comparative anatomy, which would divert your attention from the present subject. But the above-mentioned principle of my theory of Anthropogenesis viz. that Man's development is acted upon by a retardation that originates in the endocrin system, I feel impelled to substantiate by some evidence. This I shall take from the domain of pathology. Congenital malformations of our body are not rare. Now it is very remarkable that all these congenital anomalies observed in Man are the consequence of a retardation of development. The pathologic in all these cases, therefore, is to be considered as the consequence of a locally too intensive action of the normal regulator of the rate of development, which prevents a regular and full accomplishment of a morphological property or characteristic. I do not know of any instance of the opposite kind. Of the very long list of congenital anomalies in Man I could not name one that might be interpreted as a consequence of an accelerated development, i.e. through the absence or lessening of the inhibitory factor. But ..... and this is undeniably extremely interesting ..... such instances are to be found in the post-natal phase of our existence.

Briefly, let me express it thus: our endocrin system secretes inhibitive hormones (which, strictly speaking, is a contradictio in adjecto). The properties suppressed in the anthropogenesis under the influence of the historic inhibition are still present in a latent condition, and must remain

suppressed all through life. So we preserve in us some properties of our ancestors, but checked. What happens in case one or more organs of our endocrin system are diseased and in consequence the normal production of hormones disturbed? Then the inhibition ceases or weakens, and properties which had been lost in the course of ages will appear again, or functions which were retarded will develop in an abnormally rapid tempo. Do you desire instances? Man has lost his hairy skin, i.e. the growth of hair was first retarded, then suppressed. When the endocrin system is diseased, the inhibition may cease and the hairy coat reappears. One group of the human race has lost the pigment in the skin. Now with affections of the endocrin system it may reappear, the production of hormones which suppress the development of these properties is disturbed. During Anthropogenesis the jaws of Man have become relatively smaller in compass as well as in size, because fetal conditions became more and more permanent; if the endocrin system becomes diseased, the inhibition ceases and the jaws — and not infrequently the supra-orbital ridges also which are governed by similar conditions — begin to enlarge again, and become stronger. Contrary to what occurs in the other Primates the sutures of the human skull do not close after the brain has attained its definite size. They persist, their obliteration is inhibited and is established sometimes only in old age. The endocrin system becomes diseased, one of the symptoms being the possible premature closing of one or more sutures and consequent cranial malformation. This goes to show how a number of what we may call "monkey-like" properties may be present in us in a latent condition, waiting only for the disappearance of the factor that suppresses them to appear again. Let these few examples suffice to show that pathology also can furnish evidence which confirms the biological principle of anthropogenesis brought forward here. But let me add one more instance of a more functional nature. Together with the retardation of the process of Man's development the commencement of the functioning of the genital glands is delayed to a later age, i.e. its commencement is checked for some time. The endocrin system becomes diseased, and as an effect premature sexual maturity may occur, the inhibition being absent. But enough of this so productive subject, for I must draw to a conclusion, but wish first to present to you, though briefly, two new points of view.

The first is the biological relation of the human races inter se and to the other Primates. The human form, then, has arisen from a fetalization-process. But the developmental progress has not been the same with all mankind, nor has fetalization equally affected all races. For instance, the typical physiogonomy of the Mongolian race, with their flat nose-bridge and the so-called Mongol-fold of the eye, is a fetalization effect that does not occur in the white race. Yet this, as I believe, has been retarded and fetalized most, it has covered the longest distance on the path of anthropogenesis, it is the most human. But it is posible for all other races to reach

the zenith of development now occupied by the white race. The only thing required is continued progressive action in these races of the biological principle of anthropogenesis. In his fetal development the negro passes through a stage that has already become the final stage for the white man. Well then, when retardation continues in the negro too, what is still a transitional stage may for this race also become a final one.

As to the relation of Man to the other Primates, each Primate passes in his fetal life through a stage that has become the final stage for Man. Thus, in every Primate, if the retardation principle but begins to operate, lies the possibility of resemblance to the human race. To my regret the time fails me to show you in what respects this is already the case with the anthropoids.

Do you perceive how the theory propounded here alters fundamentally our view of the relation of Man to the other Primates ?

The second point of view that I still wish to discuss concerns the question of the relation of our endocrin system to the rest of our organism.

The said system is very complicated, the complex of the endocrin organs regulates the metabolism by the production of hormones. How that regulation is effected we do not know, for the terms stimulation or excitement, and inhibition or suppression are only indications of the character of the effect of the action. But with regard to its *function* I would draw a parallel between this system and the central nervous system, and *with regard to its relation to the whole organism,* a parallel between this system and the genital glands. The central nervous system also consists of a number of organs. But we have to confess to great ignorance of the associations and the co-ordinations existing between these organs. We see the effect, however, and this is always a result, but one which we have not learned to analyse. It is the same with the endocrin system. Here also a result of the functioning of a great number of organs, but the part played by each of them we do not know. Just as in the central nervous system there is not a single effect to which all the organs do not contribute, or have not contributed, directly or indirectly, so is it also in the endocrin system, the function of which is to be considered as an instance of ideal co-operation. This raises the biological significance of our endocrin system above that of a system properly so-called, it is of itself an organism, an imperium in imperio, it regulates and governs. The effect of this action is first the genesis of the form and then its maintenance. In view of this important role I would designate the whole complex of organs performing this part by the term *Endocrinon.*

Now I can draw my parallel between this and the other above-named constituent of the organism : the genital glands, or the so-called germa. For the germa is to the species as the endocrinon is to the individual. Just as the germa determined the origin of the species and ensures its maintenance, so the endocrinon serves to maintain the individual, after having governed his genesis.

Gentlemen, I have had to make severe demand upon your attention, the more since my exposition had of necessity to be compressed within a narrow compass, which may have rendered it somewhat difficult to follow. But it was no easy matter to set forth clearly, in a limited space of time and in logical context, the result of many years' speculation and investigation of such a comprehensive subject as Anthropogenesis. I might have taken some subdivision of this subject and have discussed it at more length, bringing evidence to support my assertions, but I preferred to give you an outline of my train of thought as it has gradually taken form in the course of years. I shall consider myself amply rewarded for my trouble if I have succeeded in awakening a vague conviction that the solution of the problem of Anthropogenesis must be looked for in quite a different direction from the one hitherto followed, and that the problem of the racial relations of Man is not the same as that of the development of our form.

Mathematics. — *"Allgemeine Räume und Cartesische Räume"*. (Erste Mitteilung). By K. MENGER. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of February 27, 1926).

Die Bedingungen dafür, dass ein allgemeiner topologischer oder metrischer Raum homöomorph sei mit einer Teilmenge eines Cartesischen Zahlenraumes, waren bisher fast gänzlich unbekannt. Wie es scheint, ist es erst die Dimensionstheorie, welche eine Formulierung der Tatsachen dieses Problemkreises ermöglicht, zugleich aber Methoden zu ihren Beweisen liefert [1]).

In der vorliegenden ersten Mitteilung über diesen Gegenstand wird folgender Satz bewiesen:

Jeder kompakte eindimensionale metrische Raum ist homöomorph mit einer Teilmenge des dreidimensionalen Zahlenraumes.

Die Beweismethode (welche in anschliessenden Arbeiten die Lösung wichtiger Homöomorphieprobleme für allgemeine n-dimensionale Mengen ergeben wird) entspricht den BROUWERschen Forderungen nach Konstruktivität zwar nicht vollkommen (insbesondre da das BORELsche Ueberdeckungstheorem wesentlich eingeht), aber in hohem Masse, so dass nicht nur von dem Theorem, sondern auch von dem hier dargelegten Beweis viel in die intuitionistische Mathematik übernommen werden kann. Einer in gewisser Weise entstehenden eindimensionalen Menge wird eine im R_3 entstehende Menge zugeordnet, so dass aus der Beziehung der beiderseitigen Bildungsgesetze die Homöomorphie der beiden Mengen folgt. Wir führen zunächst einige Hilfsbegriffe ein.

Sei A ein kompakter metrischer Raum. Als *finites Umgebungssystem* in A bezeichnen wir eine (abzählbare) Menge von in A abgeschlossenen Umgebungen $A_{i_1 i_2 \ldots i_n}$, welche folgendermassen entsteht:

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[1]) Schon SIERPIŃSKI wurde bei der Entdeckung des ersten Resultates dieses Fragenkreises — als er nämlich die Bedingungen dafür aufstellte, dass eine zusammenhangslose Menge mit einer Teilmenge der Zahlengeraden homöomorph sei (Fund. Math. II, S. 89), — auf jene Mengen geführt, die man, von einem allgemeinen Gesichtspunkt aus, als nulldimensional bezeichnen muss. Die Hauptschwierigkeit, welche sich der Einbettung komplizierterer Mengen in höherdimensionale Cartesische Räume entgegenstellt, nämlich die Realisierung gewisser Zusammenhangsverhältnisse, tritt bei den zusammenhangslosen Mengen allerdings noch nicht auf. Auch ein weiteres Resultat, den Beweis der MAZURKIEWICZschen Vermutung (Fund. Math. II, S. 130), dass jede stetig durchlaufbare Baumkurve mit einem ebenen Kontinuum homöomorph sei, konnte ich auf Grund von dimensions- und kurventheoretischen Sätzen herleiten (Math. Annalen 96, 1926).

1. Es gibt endlich viele Mengen $A_1, A_2, \ldots A_n$ (wir nennen sie die Mengen "des ersten Schrittes"), die zu je zweien keine inneren Punkte gemein haben.

2. Sei $A_{i_1 i_2 \ldots i_n}$ eine bereits definierte Menge "des n^{ten} Schrittes", dann gibt es endlich viele abgeschlossene Umgebungen $A_{i_1 i_2 \ldots i_n 1}, \ldots A_{i_1 i_2 \ldots i_n m}$, die sämtlich $< A_{i_1 i_2 \ldots i_n}$ sind und zu je zweien keine inneren Punkte gemein haben.

3. Es gibt ein Folge ε_n von positiven Zahlen mit $\lim \varepsilon_n = 0$, so dass die Durchmesser aller Mengen $A_{i_1 i_2 \ldots i_n}$ des n^{ten} Schrittes $< \varepsilon_n$ sind.

Als *Kern* des finiten Umgebungssystems $\{A_{i_1 i_2 \ldots i_n}\}$ bezeichnen wir die Menge aller Punkte, die für jedes n in mindestens einer System-Menge des n^{ten} Schrittes enthalten sind. Offenbar ist der Kern eines finiten Umgebungssystemes stets abgeschlossen.

Wir nennen zwei (eventuell in verschiedenen Räumen befindliche) finite Umgebungssysteme $\{A_{i_1 i_2 \ldots i_n}\}$ und $\{B_{i_1 i_2 \ldots i_n}\}$ in welchen die Mengen mit gleichen Indizes einander eineindeutig entsprechen, *homolog*, wenn folgende Bedingung erfüllt ist: Haben irgendwelche Mengen $A_{i_1 i_2 \ldots i_n}, A_{j_1 j_2 \ldots j_n}, \ldots A_{k_1 k_2 \ldots k_n}$ irgend eines n^{ten} Schrittes einen leeren Durchschnitt, dann haben die entsprechenden Mengen $B_{i_1 i_2 \ldots i_n}, B_{j_1 j_2 \ldots j_n}, \ldots B_{k_1 k_2 \ldots k_n}$ einen leeren Durchschnitt und umgekehrt. Es gilt dann folgendes

Lemma. Die Kerne homologer finiter Umgebungssysteme sind homöomorph [1]).

Man kann zunächst in folgender Weise eine eineindeutige Abbildung zwischen den Kernen A und B der homologen Systeme $\{A_{i_1 i_2 \ldots i_n}\}$ und $\{B_{i_1 i_2 \ldots i_n}\}$ angeben: Sei a Punkt von A; beim n^{ten} Schritt gehöre a etwa den Mengen $A_{i_1 i_2 \ldots i_n}, A_{j_1 j_2 \ldots j_n}, \ldots A_{k_1 k_2 \ldots k_n}$ an. Wir bezeichnen sie mit $A_1^n(a), A_2^n(a), \ldots A_{r_n}^n(a)$ und die entsprechenden Mengen $B_{i_1 i_2 \ldots i_n}, \ldots B_{k_1 k_2 \ldots k_n}$ mit $B_1^n, B_2^n, \ldots B_{r_n}^n$. Betrachten wir die Menge $B(a) = \prod_{n=1}^{\infty} \sum_{i=1}^{r_n} B_i^n$. Nach dem Cantorschen Durchschnittssatz ist $B(a)$ nicht leer. Anderseits kann $B(a)$ nicht mehr als einen Punkt enthalten; denn da die Mengen $A_1^n(a), A_2^n(a), \ldots A_{r_n}^n(a)$ den Punkt a gemein haben, ist wegen der Homologie der beiden Systeme auch der Durchschnitt der Mengen $B_1^n, B_2^n, \ldots B_{r_n}^n$ nicht leer; daher konvergieren, da die Durchmesser der Mengen B_j^n bei wachsendem n gegen Null konvergieren, die Durchmesser der Mengen $\sum_{i=i}^{r_n} B_i^n$ bei wachsendem n gegen Null. Jedem Punkt a von A wird also ein Punkt $B(a)$ von B zugeordnet, wobei verschiedenen Punkten von A verschiedene Punkte von B entsprechen und jeder

[1]) Eine starke Verallgemeinerung dieses Lemmas wird in einer folgenden Arbeit zur Lösung von Homöomorphieproblemen hinsichtlich der Mengen, *die Teilmengen halbkompakter Räume* sind, herangezogen werden. Daselbst wird auch für allgemeinere Umgebungssysteme eine Homologie definiert und der Kernbegriff verallgemeinert werden. Für Homöomorphieuntersuchungen hinsichtlich kompakter Räume reicht das Lemma in der obigen Form aus.

Punkt von B einem Punkt von A entspricht. Man überzeugt sich sofort davon, dass diese eineindeutige Abbildung der zwei Kerne in beiden Richtungen stetig, also topologisch ist.

Das Problem, zu einem vorgelegten kompakten eindimensionalen Raum A eine homöomorphe Teilmenge des R_3 anzugeben, wird durch das Lemma zurückgeführt auf das Problem, in A ein finites Umgebungssystem und im R_3 ein dazu homologes finites Umgebungssystem anzugeben. Der Lösung dieses Problems schicken wir drei weitere Hilfssätze voraus.

Hilfssatz 1 besagt im wesentlichen, *dass man zu einer bestimmt gearteten Zerlegung eines kompakten Raumes ein entsprechendes System von Polyedern innerhalb eines vorgegebenen Polyeders P angeben kann und zwar unter Erfüllung gewisser Randbedingungen, d.h. so, dass die Polyeder des Systems in vorgegebenen Strecken die Oberfläche von P berühren.* Es wird, genau gesprochen, *vorausgesetzt*: A sei ein kompakter Raum; $B_1, B_2, \ldots B_m$ seien m paarweise fremde Teilmengen von A; P sei ein Polyeder des R_3; $Q_1, Q_2, \ldots Q_m$ seien m paarweise fremde Strecken auf der Oberfläche von P.

Ferner: A sei Summe von n in A abgeschlossenen Umgebungen $A_1, A_2, \ldots A_n$, die zu je zweien keine inneren Punkte gemein haben und folgende Bedingungen erfüllen:

a. Jeder Punkt von A gehört höchstens zwei Mengen A_i an;

b. Jeder Punkt einer Menge B_k gehört genau einer Menge A_i an;

c. Jede Menge A_i hat höchstens mit einer Menge B_k Punkte gemein.

Dann wird *behauptet:* Es existieren n Polyeder $P_1, P_2, \ldots P_n < P$, die zu je zweien keine inneren Punkte gemein haben und folgende Bedingungen erfüllen:

1. P_i und P_k haben dann und nur dann Punkte und zwar genau eine Strecke (die wir Q_{ik} nennen) gemein, wenn A_i und A_k Punkte gemein haben.

2. P_i hat mit der Begrenzung von P dann und nur dann Punkte und zwar genau eine Teilstrecke von Q_k (die wir Q'_{ik} nennen) gemein, wenn A_i und B_k Punkte gemein haben.

3. Die Strecken Q_{ik} und Q'_{ik} sind paarweise fremd.

Beweis: Wir ordnen jeder Menge A_i einen Punkt p_i von P zu und wählen dabei p_i dann und nur dann auf der Oberfläche von P und zwar auf Q_k, wenn A_i und B_k Punkte gemein haben. [Wegen Voraussetzung *c* ist eine solche Wahl von Punkten möglich; natürlich können mehrere Punkte p_i auf derselben Strecke Q_k liegen]. Wir verbinden hierauf dann und nur dann, wenn A_i und A_j Punkte gemein haben, p_i und p_j durch einen Streckenzug $\overline{p_i p_j}$, der keine anderen Punkte p_k enthält und (höchstens abgesehn von seinen Endpunkten p_i und p_j) ganz im Inneren von P liegt. Die Streckenzüge $\overline{p_i p_j}$ können und wollen wir so bestimmen, dass je zweie höchstens Endpunkte mit einander gemein haben. Wir wählen sodann in jedem Streckenzug $\overline{p_i p_j}$ einen von den Endpunkten verschiedenen Punkt q_{ij} und eine kleine Strecke Q_{ij}, die mit $\overline{p_i p_j}$ blos

den Punkt q_{ij} gemein hat. Sind die Q_{ij} hinreichend klein, so kann man die Streckenzüge $\overline{p_i q_{ij}}$ und $\overline{p_j q_{ij}}$ mit je einer Kette S_{ij} bezw. S_{ji} von Simplizes $< P$ umgeben, so dass S_{ij} und S_{ji} die Strecke Q_{ij} und sonst keine Punkte gemein haben; liegt der Punkt p_i auf Q_k, dann soll jede derartige von p_i nach einem Punkt q_{ji} führende Simplizeskette S_{ij} so bestimmt werden, dass sie p_i nicht im Inneren, sondern auf der Oberfläche enthält und dass diese Oberfläche mit der Oberfläche von P genau eine Teilstrecke Q'_{ik} von Q_k gemein hat.

Sind die zum Bau der Ketten verwendeten Simplizes hinlänglich klein, dann hat eine Kette S_{ij} mit der Oberfläche von P nur dann Punkte und zwar eine Teilstrecke von Q_k gemein, wenn p auf Q_k liegt; dann haben schliesslich zwei in demselben Punkt zusammenstossende Ketten S_{ij} und S_{ik} nur in der Nachbarschaft von p_i Punkte gemein, während je zwei Ketten, die nicht in einem Punkt p_i oder in einer Strecke Q_{ij} zusammentreffen, fremd sind. Bezeichnen wir dann mit P_i die Summe der in p_i zusammentreffenden Simplizesketten S_{ij}, so erfüllen die Polyeder P_i offenbar die Forderungen des Hilfssatzes 1.

Hilfssatz 2 besagt im wesentlichen, *dass man zu einer bestimmt gearteten Kette von Polyedern eine entsprechende Zerlegung eines kompakten eindimensionalen Raumes unter Erfüllung gewisser Randbedingungen angeben kann.*

Es wird, genau gesprochen, *vorausgesetzt:* A sei ein kompakter eindimensionaler Raum; $B_1, B_2, \ldots B_m$ seien m paarweise fremde abgeschlossene nulldimensionale Teilmengen von A; P sei ein Polyeder des R_3; $Q_1, Q_2, \ldots Q_m$ seien m paarweise fremde Strecken auf der Oberfläche von P.

Ferner: Es sei $P_1, P_2, \ldots P_n$ eine Kette von Polyedern $< P$ von folgenden Eigenschaften:

1. Je zwei aufeinanderfolgende Polyeder P_i und P_{i+1} $(i = 1, 2, \ldots n-1)$ haben genau eine Strecke mit einander gemein; je zwei andere Polyeder sind fremd.

2. Der Durchschnitt der Polyederkette mit der Oberfläche von P besteht aus je einer Teilstrecke der Strecken Q_k und zwar gehöre jede solche Teilstrecke genau einem P_i an und kein P_i habe mit zwei Q_k Punkte gemein.

Dann wird *behauptet:* Es existieren n in A abgeschlossene Umgebungen $A_1 A_2, \ldots A_n$, die zu je zweien höchstens nulldimensionale Durchschnitte haben und folgenden Bedingungen genügen:

a. Jeder Punkt von A gehört höchstens zwei Mengen A_i an; je zwei fremden Polyedern P_i und P_k entsprechen fremde Mengen A_i und A_k.

b. Jeder Punkt der Menge B_k gehört der Menge A_i und nur dieser an, falls Q_k mit P_i Punkte gemein hat.

c. Jede Menge A_i hat höchstens mit einer Menge B_k Punkte gemein.

Beweis: Ist das Polyeder P_1 zur Oberfläche von P fremd, dann bezeichnen wir irgend eine in A abgeschlossene Umgebung, die zu allen Mengen B_k fremd ist und deren Begrenzung C_i nulldimensional ist, mit

A_1. Hat dagegen das Polyeder P_1 mit Q_k eine Teilstrecke gemein, dann bilden wir eine in A abgeschlossene Umgebung von B_k, die zu allen anderen B_r fremd und deren Begrenzung C_1 nulldimensional ist, und bezeichnen sie mit A_1. Wir können annehmen, es seien bereits die Mengen $A_1, A_2, \ldots A_{i-1}$ definiert; C_{i-1} sei die nulldimensionale Begrenzung von A_{i-1}. Wieder sind zwei Fälle zu unterscheiden: Wenn P_i zur Oberfläche von P fremd ist, dann bezeichnen wir mit A_i eine in $A - \sum_{j=1}^{i-1} A_j + C_{i-1}$ abgeschlossene Umgebung, die zu allen Mengen B_k fremd ist, deren Begrenzung C_i nulldimensional ist und so, dass C_{i-1} im Innern von $\sum_{j=1}^{i} A_j$ enthalten ist. Enthält dagegen P_i eine Teilstrecke von Q_k, dann bezeichnen wir mit A_i eine in $A - \sum_{j=1}^{i-1} A_j + C_{i-1}$ abgeschlossene Umgebung, deren Begrenzung nulldimensional ist und die ferner folgende Eigenschaften hat: C_{i-1} ist im Innern von $\sum_{j=1}^{i} A_j$ enthalten, B_k im Innern von A_i, während A_i zu allen anderen Mengen B_r fremd ist. Setzen wir endlich $A_n = A - \sum_{i=1}^{n-1} A_i + C_{n-1}$, dann genügen die so konstruierten Mengen $A_1, A_2, \ldots A_n$ offenbar den Forderungen von Hilfssatz 2.

Hilfssatz 3 besagt im wesentlichen, *dass man zu einem vorgelegten kompakten eindimensionalen Raum eine bestimmt geartete Zerlegung in beliebig kleine Umgebungen und gleichzeitig in einem vorgelegten Polyeder ein entsprechendes System von beliebig kleinen Polyedern unter Erfüllung beiderseitiger Randbedingungen angeben kann.*

Es wird, genau gesprochen, *vorausgesetzt:* A sei ein kompakter eindimensionaler Raum; $B_1, B_2, \ldots B_m$ seien m paarweise fremde abgeschlossene nulldimensionale Teilmengen von A; P sei ein Polyeder des R_3; $Q_1, Q_2, \ldots Q_m$ seien m paarweise fremde Strecken auf der Oberfläche von P; zwei positive Zahlen ε und η seien gegeben.

Es wird dann *behauptet:* Es gibt endlich viele in A abgeschlossene Umgebungen $A_1, A_2, \ldots A_n$, die zu je zweien höchstens nulldimensionale Durchschnitte haben und die sämtlich $< \varepsilon$ sind, so dass

a. jeder Punkt von A höchstens zwei Mengen A_i angehört,

b. jeder Punkt einer Menge B_k genau in einer Menge A_i enthalten ist,

c. jede Menge A_i mit höchstens einer Menge B_k Punkte gemein hat,

und es gibt zugleich Polyeder $P_1, P_2, \ldots P_n < P$, die sämtlich $< \eta$ sind, und so dass

1. P_i und P_j dann und nur dann Punkte und zwar eine Strecke (die wir Q_{ij} nennen) gemein haben, wenn A_i und A_j Punkte gemein haben;

2. P_i mit der Oberfläche von P dann und nur dann Punkte und zwar eine Teilstrecke von Q_k (die wir Q'_{ik} nennen) gemein hat, wenn A_i und B_k Punkte gemein hat;

3. die Strecken Q_{ik} und Q'_{ik} paarweise fremd sind.

Beweis: Nach dimensionstheoretischen Sätzen kann man zunächst A in abgeschlossene Umgebungen $A_1^\times, A_2^\times, \ldots A_n^\times$ zerlegen, die zu je zweien höchstens nulldimensionale Durchschnitte haben, die $< \varepsilon$ und überdies so klein sind, dass jede Menge $A_i^\times$ mit höchstens einer Menge B_k Punkte gemein hat und so, dass jeder Punkt von A höchstens zwei Mengen $A_i^\times$ und jeder Punkt einer Menge B_k genau einer Menge $A_i^\times$ angehört. [1]) Nach Hilfssatz 1 kann man Polyeder $P_1^\times, P_2^\times, \ldots P_n^\times < P$ angeben, so dass

1. $P_i^\times$ und $P_j^\times$ dann und nur dann Punkte und zwar genau eine Strecke (die wir $Q_{ij}^\times$ nennen) gemein haben, wenn $A_i^\times$ und $A_j^\times$ Punkte gemein haben;

2. $P_i^\times$ mit der Begrenzung von P dann und nur dann Punkte und zwar genau eine Teilstrecke von Q_k (die wir $Q_{ik}^{\times\prime}$ nennen) gemein hat, wenn $A_i^\times$ und $B_k^\times$ Punkte gemein haben;

3. die Strecken $Q_{ij}^\times$ und $Q_{ik}^{\times\prime}$ paarweise fremd sind.

Sind die Polyeder $P_i^\times$ sämtlich $< \eta$, dann erfüllen sie die Forderungen von Hilfssatz 3. Es kann aber etwa $P_i^\times > \eta$ sein. Dann wählen wir auf jeder der Strecken $Q_{ij}^\times$ und $Q_{ik}^{\times\prime}$, die auf der Oberfläche von $P_i^\times$ gelegen sind, eine Teilstrecke $< \frac{\eta}{2}$, die wir bezw. mit Q_{ij} und Q'_{ik} bezeichnen. Sodann bilden wir einen Streckenzug, der mit jeder auf der Oberfläche von $P_i^\times$ befindlichen Strecke Q_{ij} und Q'_{ik} je einen Punkt gemein hat und sonst ganz im Innern von $P_i^\times$ verläuft. Diesen Streckenzug können wir durch passende Aneinanderreihung von Simplizes ersetzen durch eine Kette $P_{i1}^\times, P_{i2}^\times, \ldots P_{in_i}^\times$ von Polyedern $< P$ und $< \eta$ mit den in Hilfssatz 2 vorausgesetzten Eigenschaften. Nach Hilfssatz 2 entsprechen ihnen Teilmengen $A_{i1}^\times, A_{i2}^\times, \ldots A_{in_i}^\times$ von $A_i^\times$. Sollten aufeinanderfolgende Mengen $A_{ik}^\times$ und $A_{ik+1}^\times$ fremd sein, so können die entsprechenden Polyeder durch kleinere zu einander fremde Polyeder ersetzt werden, die wir statt dessen mit $P_{ik}^\times$ und $P_{ik+1}^\times$ bezeichnen. Wählt man ferner auf $Q_{ij}^\times$ dieselbe Strecke Q_{ij} für $P_i^\times$ und $P_j^\times$, dann erfüllt die Gesamtheit der vorliegenden Umgebungen $A_{ik}^\times$ und der Polyeder $P_{ik}^\times$ $(i = 1, 2, \ldots n, k = 1, 2, \ldots n_i)$, wenn diese Mengen irgendwie mit $A_1, A_2, \ldots A_n$, bezw. mit $P_1, P_2, \ldots P_n$ bezeichnet werden, offenbar die Forderungen von Hilfssatz 3.

Sei nun ein kompakter eindimensionaler Raum vorgelegt. Wir geben zwei gegen Null konvergente Folgen $\{\varepsilon_n\}$ und $\{\eta_n\}$ von positiven Zahlen vor. Nach Hilfssatz 3 ist A Summe von endlich vielen abgeschlossenen Umgebungen $< \varepsilon_1$ $A_1, A_2, \ldots A_n$ mit zu je zweien höchstens nulldimensionalen Durchschnitten, so dass jeder Punkt von A höchstens zwei Mengen A_i angehört und zugleich gibt es Polyeder $P_1, P_2, \ldots P_n$,

[1]) Vgl. etwa Monatshefte f. Math. u. Phys. 34, S. 153; dass die Zerlegung auch so eingerichtet werden kann, dass jeder Punkt einer der nulldimensionalen Mengen B_k genau einer Menge $A_i^\times$ angehört, folgt daraus, dass man die a. a. O. angegebene Konstruktion ausschliesslich mit Umgebungen durchführen kann, deren Begrenzungen zu allen B_k fremd sind.

In den Beweis des dimensionstheoretischen Satzes geht der BOREL'sche Überdeckungssatz wesentlich ein.

$< \eta_1$, so dass P_i und P_k dann und nur dann Punkte und zwar eine Strecke Q_{ik} gemein haben, wenn A_i und A_k Punkte gemein haben, während die Strecken Q_{ik} paarweise fremd sind.

Auf jede der Mengen A_i und das zugehörige P_i kann nun wieder der Hilfssatz 3 angewendet werden und es können Mengen $A_{i_1 i_2} < \varepsilon_2$ und Polyeder $P_{i_1 i_2} < \eta_2$ in entsprechenden Lagen zu einander und zu den Begrenzungen von A_{i_1} und P_{i_1} bestimmt werden. Durch fortgesetzte Anwendung des Hilfssatzes 3 kann offenbar ein finites Umgebungssystem $\{A_{i_1 i_2 \ldots i_n}\}$ und gleichzeitig ein finites System von Polyedern $\{P_{i_1 i_2 \ldots i_n}\}$ erzeugt werden, so dass alle Umgebungen des n^{ten} Schrittes $< \varepsilon_n$ und alle Polyeder des n^{ten} Schrittes $< \eta_n$ sind.

Die beiden Systeme sind homolog, denn zwei Polyeder irgend eines n^{ten} Schrittes haben dann und nur dann Punkte gemein, wenn die entsprechenden Umgebungen Punkte gemein haben, während je drei Umgebungen und je drei Polyeder desselben Schrittes fremd sind. Der Kern A des Umgebungssystems ist also dem Lemma zufolge homöomorph mit dem im R_3 befindlichen Kern des Polyedersystems, womit unser Satz bewiesen ist.

Eine Analyse des dargelegten Beweises gibt zu einigen Bemerkungen Anlass. Erstens spielen die Zusammenhangsverhältnisse der Mengen $A_{i_1 i_2 \ldots i_n}$ bei der ganzen Konstruktion gar keine Rolle. Unzusammenhängenden Umgebungen können durch Hilfssatz 1 Polyeder und Polyedern können durch Hilfssatz 2 unzusammenhängende Umgebungen zugeordnet werden. Zweitens spielt die Kompaktheit keine wesentliche Rolle und lässt sich vermutlich bei ganz leichter Modifikation der Konstruktion durch die Annahme ersetzen, dass A Teilmenge eines kompakten Raumes sei. Drittens geht die Eindimensionalität von A wesentlich blos durch Verwendung folgender Tatsache in den Beweis ein: Zu jeder eindimensionalen Menge existiert ein finites Umgebungssystem, so dass je zwei Umgebungen des gleichen Schrittes höchstens nulldimensionale Durchschnitte haben. Dass derartige Systeme existieren, in welchen überdies jeder Punkt höchstens zwei Umgebungen desselben Schrittes angehört, erleichtert die Konstruktion, geht aber nicht zur Gänze in dieselbe ein.

Von Sätzen, die mit der vorliegenden Arbeit zusammenhängen und demnächst veröffentlicht werden sollen, seien folgende erwähnt: Die notwendigen und hinreichenden Bedingungen dafür, dass ein eindimensionaler Raum mit einer Teilmenge der Ebene homöomorph sei; ferner die Angabe einer umfassendsten eindimensionalen Menge, nämlich einer Kurve des R_3, welche von jedem kompakten eindimensionalen Raum ein topologisches Bild als Teil enthält. Vor allem aber können die sämtlichen hier besprochenen Verhältnisse auf höherdimensionale Mengen und Zahlenräume ausgedehnt werden und führen dann zum Beweise des topologischen Fundamentalsatzes: *Notwendig und hinreichend dafür, dass ein kompakter metrischer Raum homöomorph sei mit einer Menge eines Zahlenraumes, ist, dass er eine endliche Dimension besitze.*